

ANALYTIC GEOMETRY

MASON AND HAZARD

Quad. formula

$$-b \pm \sqrt{b^2 - 4ac}$$

$$2a$$

$$x =$$

[REDACTED]

[REDACTED]

[REDACTED]

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TEXTBOOKS IN MATHEMATICS

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ANALYTIC GEOMETRY

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PREFACE

The purpose of this book is to develop the subject matter of an introduction to analytic geometry in such a manner that the student will acquire an effective method of attack upon the problems presented.

The only departure from the usual order of presentation is that of placing the work on the straight line ahead of the general discussion of equation and locus. The student is usually already familiar with the relation of the first degree equation and the straight line from his work on graphing in elementary algebra. Experience seems to show that the treatment of the straight line tends to clarify the work on the more general correspondence between equations and geometric loci.

The lists of exercises furnish ample material for drill and at the same time contain exercises which encourage the student to get away from mere substitution in formulas. The miscellaneous lists of exercises are meant to serve two purposes. In the first place they offer material for a review of the chapter in which they occur, and in the second place they include exercises of sufficient difficulty to enable the teacher to test the intellectual mettle of his better students.

The authors desire to express appreciation to their colleagues at Purdue University for many helpful suggestions; in particular to Professor William Marshall for encouragement in undertaking the task and to Professors Laurence Hadley and William A. Zehring for a critical reading of the manuscript. They wish to acknowledge their indebtedness also to the editor, Professor Robert D. Carmichael, for his advice and constructive criticisms.

T. E. M.
C. T. H.

PURDUE UNIVERSITY

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ANALYTIC GEOMETRY

PLANE ANALYTIC GEOMETRY

CHAPTER I

THE POINT

1. Definition. Analytic geometry is a branch of mathematics in which one studies geometry by means of algebra. A correspondence is set up between a geometric locus and one or more equations, and properties of the locus are determined from the equations. In order to connect algebra and geometry use is made of a coördinate system which furnishes a means for locating points in a plane or in space.

2. Rectangular coördinates. In the coördinate system employed most frequently two straight lines intersecting at right angles* are used as lines of reference. A point is located in the plane of these lines by giving its perpendicular distance and direction from each of them. The distances with signs indicating directions are called the **coördinates** of the point. The lines from which distances are measured are called the **axes of coördinates**, or briefly **axes**. The point of intersection of the axes is called the **origin of coördinates**, or the **origin**. In Fig. 1 the line XOX' is called the x -axis, and the line YOY' is called the y -axis. Dis-

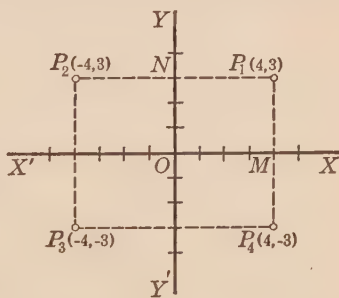


FIG. 1

* Two lines not at right angles may be used, but the resulting algebra is, in general, more complicated than in the case of lines at right angles. Any system of coördinates using two intersecting lines as axes is called Cartesian after René Descartes (1596-1650), a French mathematician.

tances measured to the right of the y -axis are usually called positive and those measured to the left negative. Distances measured upward from the x -axis are usually called positive and those measured downward negative. The choice of positive direction is a matter of convenience and may change from problem to problem. That coördinate which indicates the distance of a point to the right or to the left of the y -axis is called the **abscissa**, or **x -coördinate** of the point; and that coördinate which indicates the distance above or below the x -axis is called the **ordinate**, or the **y -coördinate** of the point. The position of a point is indicated by writing its coördinates in a parenthesis, thus: (abscissa, ordinate). For example, the point P_1 in Fig. 1 is (NP_1, MP_1) , or $(4, 3)$; the point P_2 is $(-4, 3)$; etc. Locating a point when its coördinates are given is called **plotting** the point. It is convenient to locate a point, say P_1 , by laying off $OM = x$ and then $MP_1 = y$.

3. Directed line segments. A line segment is that part of a line which is terminated by two given points on it. A **directed line segment** is a line segment to which either a positive or a negative direction has been assigned. If a positive direction is assigned to the line segment drawn from A to B (read AB), the opposite direction, from B to A , is negative, and vice versa. In either case



FIG. 2

$$BA = -AB \quad \text{or} \quad AB = -BA.$$

4. Length of a line segment parallel to a coördinate axis. Let P_1P_2 be a line segment parallel to the x -axis and consider three cases as shown in the accompanying figures. Let M_1 and M_2 be points on the x -axis whose abscissas are the same as those of P_1 and P_2 , respectively. In each of the three figures

$$P_1P_2 = M_1M_2.$$

In Fig. 3, a , $M_1M_2 = OM_2 - OM_1 = x_2 - x_1$.

In Fig. 3, b , $M_1M_2 = M_1O + OM_2 = OM_2 - OM_1 = x_2 - x_1$.

In Fig. 3, c , $M_1M_2 = M_1O - M_2O = OM_2 - OM_1 = x_2 - x_1$.

Thus in each of the three cases we can say that the length of P_1P_2 is equal to the abscissa x_2 of the end point P_2 minus the abscissa x_1 of the beginning point P_1 .

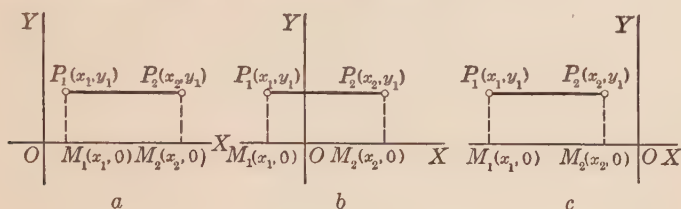


FIG. 3

In a similar manner it can be shown that the length of a line segment Q_1Q_2 parallel to the y -axis is equal to the ordinate of the end point Q_2 minus the ordinate of the beginning point Q_1 .

What is the change in temperature if the height of the mercury column in a thermometer changes from the 12° mark to the 20° mark? from -12° to 20° ? from -12° to -8° ? from 8° to -20° ?

5. Areas of triangles and polygons. The following example illustrates a method of finding the area of a triangle when the coördinates of the vertices are given.

EXAMPLE. Find the area of the triangle whose vertices are $A(1, 7)$, $B(9, 2)$, and $C(5, -3)$.

Draw through the vertices of the triangle the auxiliary lines MN , AM , and BN , as shown in the figure. Then

area $\triangle ABC$ = area trapezoid $AMNB$

$$\begin{aligned}
 &= \text{area } \triangle AMC + \text{area } \triangle BNC \\
 &= \frac{MA + NB}{2} \cdot MN - \frac{MA \cdot MC}{2} - \frac{CN \cdot NB}{2} \\
 &= \frac{10 + 5}{2} \cdot 8 - \frac{10 \cdot 4}{2} - \frac{4 \cdot 5}{2} = 30.
 \end{aligned}$$

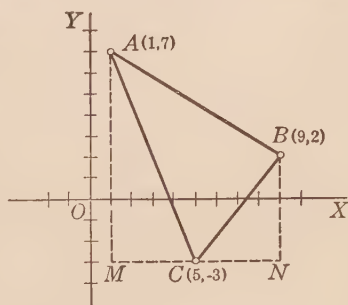


FIG. 4

The position of the necessary auxiliary lines will depend upon the location of the vertices. The student will soon learn what lines are needed to produce trapezoids and triangles with bases and altitudes parallel to the axes. There are usually several ways in which this can be done. The area of any polygon may be found in a similar manner.

EXERCISES

1. Plot the points $(2, 5)$; $(-3, -1)$; $(6, 0)$; $(0, -4)$; $(0, 0)$; $(-2.5, 1.25)$; $(\sqrt{2}, 1 - \sqrt{2})$.
2. Plot the following points on one diagram: $(-3, 9)$; $(0, 3)$; $(\frac{1}{2}, 2)$; $(1, 1)$; $(\frac{3}{2}, 0)$; $(3, -3)$; $(5, -7)$.
3. Plot the following points and connect them by a smooth curve in the order of increasing abscissas: $(-2, -1)$; $(-\frac{3}{2}, 2\frac{1}{8})$; $(-1, 3)$; $(-\frac{1}{2}, 2\frac{3}{8})$; $(0, 1)$; $(\frac{1}{2}, -\frac{3}{8})$; $(1, -1)$; $(2, 3)$.
4. Plot the following points on one diagram: (a, b) ; $(-a, b)$; $(-a, -b)$; $(a, -b)$.
5. Find the length of the line segment drawn from

(a) $(-5, 0)$ to $(7, 0)$.	(d) $(10, -3)$ to $(0, -3)$.
(b) $(0, 4)$ to $(0, -11)$.	(e) $(1, 6)$ to $(1, 0)$.
(c) $(2, -9)$ to $(2, -1)$.	(f) (a, b) to $(a + c, b)$.
6. Find the area of the triangle whose vertices are

(a) $(2, 4)$, $(7, 2)$, and $(5, 8)$.	(d) $(-3, -4)$, $(5, -2)$, and $(8, 6)$.
(b) $(3, 1)$, $(6, -2)$, and $(10, 4)$.	(e) $(-4, 5)$, $(-2, -6)$, and $(6, 2)$.
(c) $(0, 0)$, $(3, 7)$, and $(8, -2)$.	(f) $(2, 0)$, $(9, 0)$, and $(-3, 10)$.
7. Find the area of the quadrilateral whose vertices are

(a) $(-7, 3)$, $(9, 5)$, $(4, 10)$, and $(-5, 8)$;	(b) $(-6, -6)$, $(4, -2)$, $(8, 5)$, and $(-10, 7)$.
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8. Find the area of the polygon whose vertices are $(2, 0)$, $(7, 0)$, $(8, 5)$, $(5, 9)$, and $(0, 3)$.

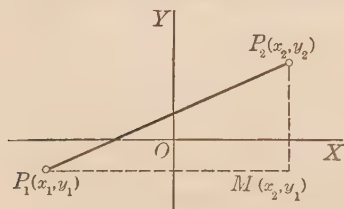


FIG. 5

6. Distance between any two points. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points in the plane (Fig. 5). Join P_1 and P_2 by a straight line. Through P_1 draw a line parallel to the

x -axis and through P_2 draw a line parallel to the y -axis. These lines intersect in $M(x_2, y_1)$. The triangle P_1MP_2 is a right triangle. Hence

$$\overline{P_1P_2}^2 = \overline{P_1M}^2 + \overline{MP_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

and

$$\overline{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \quad (1)$$

EXAMPLE. Find the distance between $(-3, -6)$ and $(5, -2)$.

Choosing $(-3, -6)$ for (x_2, y_2) and using equation (1), we have distance

$$d = \sqrt{[-3 - 5]^2 + [-6 - (-2)]^2} = \sqrt{64 + 16} = \sqrt{80} = 4\sqrt{5}.$$

The positive sign is used before the radical since we are seeking the numerical measure of the distance between the points without regard to the direction in which it is measured. It is also to be noted that $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$, and hence either point might have been chosen for (x_2, y_2) .

7. Point of division. The coördinates of a point P on the line segment P_1P_2 (Fig. 6) such that $P_1P/PP_2 = r_1/r_2$ can be found by the following method. Through P_1, P , and P_2 , respectively, draw P_1M_1, PM , and P_2M_2 perpendicular to the x -axis. We have

$$\frac{P_1P}{PP_2} = \frac{M_1M}{MM_2} = \frac{x - x_1}{x_2 - x}.$$

But $P_1P/PP_2 = r_1/r_2$. Hence

$$\frac{x - x_1}{x_2 - x} = \frac{r_1}{r_2}.$$

Solving for x , we have

$$x = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}. \quad (2)$$

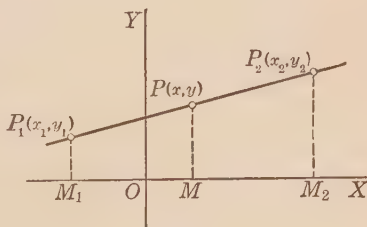


FIG. 6

Similarly, by drawing lines through P_1, P , and P_2 perpendicular to the y -axis, we can show that

$$y = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}. \quad (3)$$

In case the ratio r_1/r_2 is negative, the line segments P_1P and PP_2 must be opposite in sign, that is, must be measured in opposite directions. In this case P is not between P_1 and P_2 . If the ratio r_1/r_2 is numerically greater than 1, the segment P_1P is numerically greater than PP_2 and P lies beyond P_2 on P_1P_2 produced. If the ratio is numerically less than 1, then P lies beyond P_1 on P_2P_1 produced. In either case it can be shown that the coördinates (x, y) of the point of division are given by the equations (2) and (3).

An important particular case of finding the point of division occurs if P is the **midpoint** of P_1P_2 . In this case $r_1 = r_2$ and the above equations (2) and (3) become

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}. \quad (4)$$

EXAMPLE. Find the points which divide the line segment joining $(-3, 5)$ and $(7, -2)$ into three equal parts.

Call P_1 the point $(-3, 5)$ and P_2 the point $(7, -2)$. Then for the point of division nearest P_1 we shall have P_1P equal to one third and PP_2 equal to two thirds of the segment. Hence $r_1/r_2 = \frac{1}{2}$. Using equations (2) and (3),

$$x = \frac{1 \cdot 7 + 2 \cdot (-3)}{1 + 2} = \frac{1}{3}, \quad y = \frac{1 \cdot (-2) + 2 \cdot 5}{1 + 2} = 2\frac{2}{3}.$$

For the other point of division $r_1/r_2 = 2/1$. Hence

$$x = \frac{2 \cdot 7 + 1 \cdot (-3)}{1 + 2} = 3\frac{2}{3}, \quad y = \frac{2 \cdot (-2) + 1 \cdot 5}{1 + 2} = \frac{1}{3}.$$

EXERCISES

1. Find the distance between (a) $(5, -2)$ and $(-1, 6)$; (b) $(-3, -7)$ and $(9, 1)$; (c) $(0, 0)$ and $(-6, 3)$; (d) $(\sqrt{2}, 1)$ and $(\sqrt{3}, -\sqrt{6})$.

2. Find the distance between (a) $(a + b, c + d)$ and $(a - b, c - d)$; (b) $(a, a + b)$ and $(c, c + b)$.

3. Find the perimeter of the triangle whose vertices are $(-4, 3)$, $(5, 3)$, and $(11, -5)$.

4. Show that the triangle whose vertices are $(0, 1)$, $(-2, 0)$, and $(2, -3)$ is a right triangle.

5. Show that the triangle whose vertices are $(-1, 3)$, $(7, -5)$, and $(8, 4)$ is isosceles.

6. Show that the points $(-2, 0)$, $(5, -1)$, and $(2, 8)$ lie on a circle whose center is $(2, 3)$.

7. Find the midpoint of the line segment whose ends are
(a) $(-3, 9)$ and $(7, -1)$; (b) $(1 - \sqrt{2}, 2)$ and $(1 + \sqrt{2}, 2)$;
(c) (a, c) and (b, d) .

8. One end of a diameter of a circle with center at $(3, -2)$ is $(4, 4)$. Find the coordinates of the other end.

9. Find the lengths of the medians of the triangle whose vertices are $(-3, -5)$, $(9, 1)$, and $(-3, 7)$.

10. Find the point on each median in Exercise 9 which is at a distance from the vertex equal to two thirds of the length of the median.

11. A line is drawn across the triangle $A(-4, -3)$, $B(8, 0)$, and $C(6, 12)$ parallel to the base AB , and crossing the side AC at $(0, 3)$. Find the point where it crosses BC .

12. Find the coordinates of a point C to which the line drawn from $A(-1, 1)$ to $B(2, 6)$ must be extended (a) if $AC/AB = 5$;
(b) if $BC/AB = 5$.

13. Find a point on the line through $(1, -2)$ and $(5, 6)$ three times as far from the first point as from the second. (Two solutions.)

14. The line segment drawn from $(-5, 1)$ to $(1, 3)$ is extended 10 units. Find the coordinates of the end point.

15. Find the point on the y -axis which is equidistant from $(1, -2)$ and $(3, 4)$.

16. Find the point whose ordinate equals its abscissa and which is equidistant from $(0, \frac{4}{3})$ and $(4, -\frac{4}{3})$.

17. Find the point equidistant from $(1, 3)$, $(4, 2)$, and $(5, 1)$.

18. The ends of a chord of a circle whose radius is $5\sqrt{2}$ are $(3, -4)$ and $(5, 2)$. Find the center of the circle. (Two solutions.)

19. The base of an isosceles triangle joins the points $(0, 0)$ and $(8, -6)$. The altitude is 5 units. Find the coordinates of the vertex. (Two solutions.)

20. Two vertices of an equilateral triangle are $(-2, -1)$ and $(5, 0)$. Find the third vertex. (Two solutions.)

8. Slope of a line. It is convenient to give the direction of a line by means of the tangent of the angle θ (Fig. 7, *a* and *b*), $0 \leq \theta < 180^\circ$, measured from the positive direction of the x -axis to the line. The angle θ is called the **angle of inclination**, and its tangent, usually denoted by m , is called the **slope** of the line.

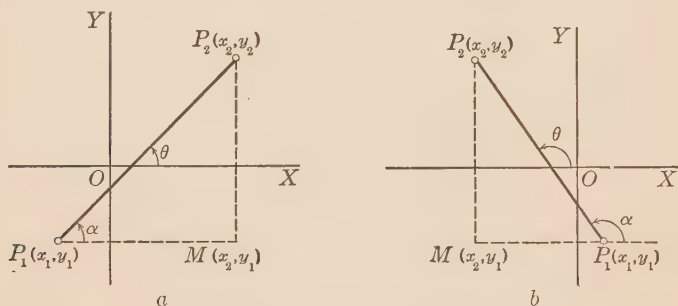


FIG. 7

The slope of the line segment P_1P_2 in Fig. 7 (*a* and *b*) is easily expressed in terms of the coördinates of P_1 and P_2 by drawing the triangle P_1MP_2 . In either figure the slope m of the line is given by

$$m = \tan \theta = \tan \alpha = \frac{MP_2}{P_1M},$$

or

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \quad (5)$$

Since

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2},$$

it is evident that the slope of a line may be interpreted as the ratio of the change in vertical distance to the change in horizontal distance as a point moves along the line in either direction.

EXAMPLE. Find the slope of the line through $(2, 3)$ and $(5, -4)$.

Using equation (5), we have

$$m = \frac{-4 - 3}{5 - 2} = -\frac{7}{3}.$$

9. Parallel lines. Perpendicular lines. Two lines which are parallel have equal angles of inclination and therefore have equal slopes, and conversely.

If two lines, having slopes $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$, meet at right angles, as in Fig. 8, then $\theta_2 = 90^\circ + \theta_1$ and

$$m_2 = \tan \theta_2 = \tan (90^\circ + \theta_1) \\ = -\cot \theta_1 = -\frac{1}{\tan \theta_1},$$

or
$$m_2 = -\frac{1}{m_1}.$$

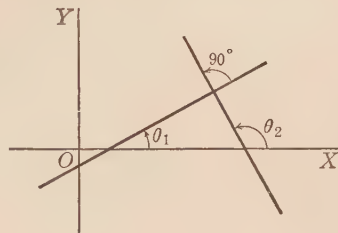


FIG. 8

Hence two lines which are perpendicular have slopes which are negative reciprocals each of the other, and conversely.

EXAMPLE. Find the slope of a line perpendicular to the line segment joining the points (2, 3) and (5, -4).

The slope of the line segment joining the given points was found in the example at the end of the last section to be $-\frac{7}{3}$. The negative reciprocal of this number is $\frac{3}{7}$. Hence any line with slope $\frac{3}{7}$ will be perpendicular to the line through (2, 3) and (5, -4).

10. Angle of intersection of two lines. Let $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$ be the slopes of the lines l_1 and l_2 (Fig. 9), respectively. Let θ be the angle measured from the line l_1 to the line l_2 . Then $\theta = \theta_2 - \theta_1$ and

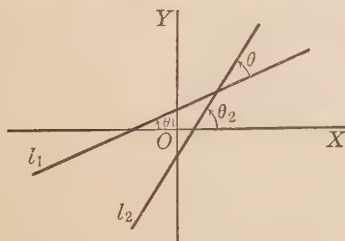


FIG. 9

$$\tan \theta = \tan (\theta_2 - \theta_1) \\ = \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_1 \tan \theta_2},$$

or
$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

It will be observed that the expression for $\tan \theta$ gives the tangent of that angle θ which has the line with slope m_1 for its initial side and the line with slope m_2 for its terminal side, if θ is measured in the counter-clockwise direction.

EXAMPLE. Given a line through $A(1, 5)$ and $B(4, 1)$, and a line through $C(2, 6)$ and $D(3, -1)$, intersecting at E , find the tangent of the angle DEB .

Plotting the points, it is seen that DE is the initial side. Hence

$$m_1 = \frac{6 - (-1)}{2 - 3} = -7 \quad \text{and} \quad m_2 = \frac{5 - 1}{1 - 4} = -\frac{4}{3},$$

$$\text{and} \quad \tan \angle DEB = \frac{-\frac{4}{3} - (-7)}{1 + \frac{28}{3}} = \frac{17}{31}.$$

EXERCISES

1. Find the slope of the line segment joining

- (a) $(3, -4)$ and $(5, 2)$. (c) $(1, 0)$ and $(-2, 9)$.
 (b) $(0, 0)$ and $(3, 4)$. (d) $(-2, 3)$ and $(6, 3)$.

2. Draw the line which passes through

- (a) $(0, 2)$ with slope $\frac{1}{3}$. (c) $(3, -4)$ with slope 4.
 (b) $(3, 0)$ with slope $-\frac{2}{3}$. (d) $(-1, -5)$ with slope -3 .

3. Find the slopes of the sides of the triangle whose vertices are $(3, 3)$, $(10, 3)$, and $(0, -3)$.

4. Show that $(-4, -2)$, $(0, 1)$, and $(20, 16)$ lie on a straight line.

5. Show that the triangle $(8, 1)$, $(5, 10)$, and $(-1, -2)$ is a right triangle.

6. Show that $(0, 0)$, $(1, -2)$, $(4, 3)$, and $(3, 5)$ are the vertices of a parallelogram.

7. Show that $(-2, 2)$, $(2, -6)$, $(8, -3)$, and $(4, 5)$ are the vertices of a rectangle.

8. Show that $(-2, 5)$, $(5, 4)$, $(4, -3)$, and $(-3, -2)$ are the vertices of a square.

9. Find the acute angle between the line joining $(-1, 4)$ and $(3, 2)$, and a line whose slope is 3.

10. Find the acute angle between the diagonals of the parallelogram in Exercise 6.

11. Find the angles of the triangle whose vertices are $(-1, 1)$, $(1, -4)$, and $(16, 2)$.

12. Find the angles of the triangle whose vertices are $(-3, 1)$, $(8, 1)$, and $(7, 6)$.

13. Show that $(-1, -3)$, $(8, 3)$, $(3, 4)$, and $(0, 2)$ are the vertices of an isosceles trapezoid.

14. Three vertices of a parallelogram are $(-1, 6)$, $(2, 1)$, and $(4, 4)$. Find a fourth vertex. (Three solutions.)

15. The angle between two lines is 45° . The slope of one of the lines is $\frac{3}{2}$. What is the slope of the other line? (Two solutions.)

16. The slope of the hypotenuse of an isosceles right triangle is -3 . What are the slopes of the two legs? 2

17. Two vertices of an equilateral triangle are $(0, 0)$ and $(6, 6)$. Find the slopes of the three sides.

11. **Polar coördinates.** Another method of locating a point P in a plane is obtained by giving its distance OP (Fig. 10) from a fixed point O and the angle MOP which OP makes with a fixed line through O . The line OM is called the **polar axis**, the line OP is called the **radius vector** of the point P , and the point O is called the **pole**. The polar coördinates of P are designated by (ρ, θ) , where ρ is the distance OP and θ is

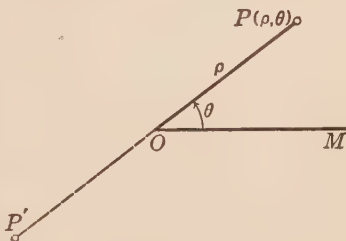


FIG. 10

the vectorial angle MOP . The distance ρ is measured from the pole and is considered positive if measured along the terminal side of θ , and is considered negative if measured in the opposite direction. Thus in Fig. 10 the point P could be located by giving θ the value of the positive angle MOP' and making ρ negative; or by giving θ the value of the negative angle MOP and making ρ positive. In the polar-coördinate system there are many sets of coördinates of the same point, but if the coördinates of a point are given there is only one point which has those coördinates.

12. **Relations between rectangular and polar coördinates.** If we make the pole of the polar system coincide with the

origin of the rectangular system, and place the polar axis on the positive x -axis, it is easy to find relations between the

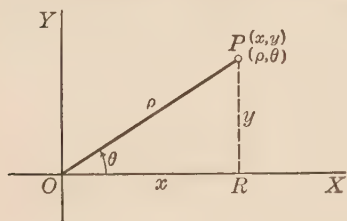


FIG. 11

rectangular coördinates (x, y) and the polar coördinates (ρ, θ) of any point P . From the triangle OPR we have

$$x = \rho \cos \theta, \quad y = \rho \sin \theta,$$

and

$$\rho = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x}.$$

EXERCISES

1. Plot the points $(2, 30^\circ)$; $(2, -30^\circ)$; $(-2, 30^\circ)$; $(-2, -30^\circ)$.

2. Plot the points $(5, \pi/4)$; $(10, \pi/2)$; (a, π) ; $(0, \theta)$; $(a, 2n\pi)$, where n is any integer.

3. A regular hexagon with sides each 4 units in length has its center at the pole and one vertex on the polar axis. What are the polar coördinates of its vertices?

4. Give at least three more distinct pairs of polar coördinates representing the point $(1, \pi/3)$.

5. Give the rectangular coördinates of each of the points in Exercises 1 and 2.

6. Plot on one diagram $(1, 0)$; $(2\sqrt{3}/3, \pi/6)$; $(\sqrt{2}, \pi/4)$; $(2, \pi/3)$. Show that these points lie on a straight line perpendicular to the polar axis.

7. Plot on one diagram and connect with a smooth curve in the order of increasing vectorial angles $(4, 0)$; $(2 + 2\sqrt{3}, \pi/6)$; $(4\sqrt{2}, \pi/4)$; $(2 + 2\sqrt{3}, \pi/3)$; $(4, \pi/2)$; $(2\sqrt{3} - 2, 2\pi/3)$; $(0, 3\pi/4)$; $(2 - 2\sqrt{3}, 5\pi/6)$; $(-4, \pi)$.

8. Give the polar coördinates of the points whose rectangular coördinates are $(1, 1)$; $(0, 3)$; $(-1, -\sqrt{3})$; $(-3, 0)$; $(4, -4)$; $(-6, 8)$; (h, k) .

9. Find the distance between $(6, 2\pi/3)$ and $(8, \pi/3)$. Check the answer by using rectangular coördinates.

10. Show that $(4, 5\pi/6)$, $(12 - 4\sqrt{3}, \pi/2)$, and $(12, \pi/3)$ lie on a straight line.

11. Find the area of the triangle whose vertices are $(0, 0)$, $(10, \pi/6)$, and $(12, \pi/3)$.

12. Find the area of the triangle whose vertices are (a) $(4, \pi/3)$, $(6, 2\pi/3)$, and $(8, 4\pi/3)$; (b) $(10, \pi/6)$, $(8, \pi/2)$, and $(5, 5\pi/6)$.

13. Applications to elementary geometry. Choice of axes.

The methods of analytic geometry will be found more powerful in the attack upon many of the problems of geometry than the methods which the student has thus far employed. Analytic geometry not only simplifies the proofs of many of the propositions with which we are familiar, but enables us to attack successfully problems which we could handle in elementary geometry only with great difficulty, or not at all. That the tools already developed — the formulas for distance, point of division, and slope — will aid in solving problems of geometry is illustrated by the exercises at the end of this section.

The properties of a geometric figure depend upon the relations of the parts and not upon the particular position in which the figure is drawn. Hence the properties of any geometric figure are independent of the way in which the axes are chosen. In the proof of geometric properties of figures it will, in general, be possible to choose the axes in more than one way. The axes should be chosen in the way which gives the simplest algebra. If we have a vertex of the figure at the origin and some line of the figure for an axis we shall usually have the best choice of axes. Problems differ, however, and no general rule is without exception. Much must be left to the ingenuity of the student.

EXAMPLE. The line joining the midpoints of the nonparallel sides of a trapezoid is equal to one half of the sum of the parallel sides and is parallel to them.

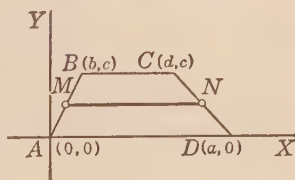


FIG. 12

Choose a base of the trapezoid for the x -axis and one end of that base for the origin, as in the figure. The coördinates of the four vertices may be taken

as indicated. The midpoint of AB is $(b/2, c/2)$ and of CD is $((a+d)/2, c/2)$. The slope of MN is found to be zero, and MN is therefore parallel to the x -axis and hence is parallel to AD and BC . Since MN is parallel to the x -axis, its length is

$$MN = \frac{a+d}{2} - \frac{b}{2} = \frac{a+d-b}{2}.$$

But $AD = a$ and $BC = d - b$. Hence

$$MN = \frac{1}{2}(AD + BC).$$

EXERCISES

Prove analytically:

1. The diagonals of a rectangle are equal.
2. The diagonals of a square are perpendicular to each other.
3. The diagonals of a parallelogram bisect each other.
4. The line drawn from the vertex of the right angle of a right triangle to the middle point of the hypotenuse is equal to one half of the hypotenuse.
5. The line joining the midpoints of two sides of a triangle is equal to one half of the third side and is parallel to it.
6. Two medians of an isosceles triangle are equal.
7. The diagonals of an isosceles trapezoid are equal.
8. Lines joining the midpoints of the sides of a quadrilateral taken in order form a parallelogram.
9. Lines joining the midpoints of the opposite sides of a quadrilateral bisect each other.
10. If a median is drawn from the vertex to the base of a triangle the sum of the squares of the other two sides is equal to twice the square of one half of the base plus twice the square of the median.
11. The sum of the squares of the four sides of a quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the line joining the midpoints of the diagonals.
12. The three lines joining the alternate midpoints of the sides of a regular hexagon form an equilateral triangle. (Use polar coördinates.)
13. The area of a triangle whose vertices are (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) is $\frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$.

CHAPTER II

THE STRAIGHT LINE

14. Correspondence between geometric figure and equation. The coördinate systems described in Chapter I can be used to set up a correspondence between equations and certain geometric figures. In this chapter is discussed the correspondence between straight lines and first degree equations in two variables. Two problems present themselves:

1. *Given the line to find the corresponding equation.*
2. *Given the equation to locate the corresponding line.*

The nature of these two problems will be illustrated by examples.

EXAMPLE 1. Find the equation which expresses the relation between the abscissa and the ordinate of any (every) point on the line through the point $(3, 1)$ and with the slope $\frac{1}{2}$.

Let $P(x, y)$ be any point on the given line. The slope of the line segment joining P to $(3, 1)$ is $\frac{1}{2}$. The slope of this line segment is also $(y - 1)/(x - 3)$. Hence

$$\frac{y - 1}{x - 3} = \frac{1}{2},$$

or
$$x - 2y = 1,$$

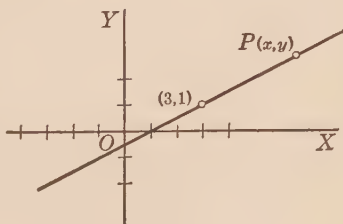


FIG. 13

is the equation expressing the relation between the abscissa and the ordinate of any point on the line. It is called the equation of the line. The equation is of the first degree. It will be shown in Sec. 21 that to every straight line corresponds an equation of the first degree, and conversely.

From the foregoing example we see that the coördinates x and y of every point (x, y) on the given line satisfy the

equation corresponding to the line. This suggests the method of attack on problem 2, namely, that of locating a line when its equation is given. Pairs of numbers which satisfy an equation of the first degree in two variables are taken as abscissas and ordinates of points. These points when plotted lie on a straight line which corresponds to the given equation. This straight line is called the **graph** of the equation. *It is the line which passes through all the points, and only the points, whose coördinates satisfy the equation.*

EXAMPLE 2. Graph the line represented by the equation $2x - y = 6$.

The following pairs of numbers will be found to be solutions of the given equation :

$x =$	-2	-1	0	1	2	3	4	5	6
$y =$	-10	-8	-6	-4	-2	0	2	4	6

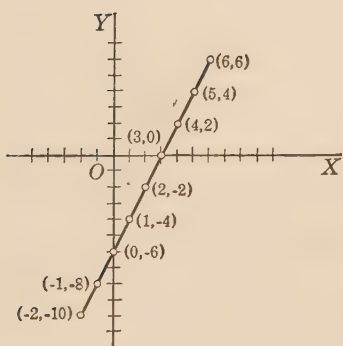


FIG. 14

Each pair of numbers represents a point on the required line. It is obviously impossible to write down all the number pairs which are solutions of the equation. If the points represented by these number pairs are plotted, they show the position of the line corresponding to the equation. Since the graph is a straight line, it is necessary to plot but two points, for example, the points $(0, -6)$ and $(3, 0)$.

EXERCISES

1. Find and plot at least five points which lie on the line whose equation is given. Draw the line in each case.

(a) $x + y = 8$.

(c) $7x + 5y + 2 = 0$.

(b) $2x - 3y = 6$.

(d) $x - 3y = 0$.

2. Find the points where the following lines cross the axes and draw the line in each case:

(a) $3x - 4y = 12$.

(b) $2x - 5y + 9 = 0$.

3. Which of the following lines pass through (1) the origin; (2) the point (1, 1); (3) the point (3, -2)?

(a) $x + 2y = 3$.

(c) $3x + 2y = 5$.

(b) $4x - y = 0$.

(d) $3y + 2x = 0$.

4. Write the equation of the line through (4, -1) and having the slope $\frac{1}{2}$.

5. Write the equation of the line through (-1, 3) and (2, 6).

6. Write the equation of the line through (2, 5) parallel to the line determined by (0, -4) and (6, 4).

7. Write the equation of the line through (2, 5) and perpendicular to the line determined by (0, -4) and (6, 4).

8. Write the equations of the two lines parallel to the x -axis and numerically 3 units distant from it.

9. Write the equations of the two lines parallel to the y -axis and numerically 5 units distant from it.

10. What line is represented by the equation (a) $x = 0$? (b) $y = 0$?

11. What is the equation of the line which bisects (a) the first and third quadrants? (b) the second and fourth quadrants?

12. Write the equation of the locus of points equidistant from (-3, 5) and (7, 1).

15. Forms of equations of lines. The method of deriving the equation of a straight line depends upon the geometric conditions which are given to describe the line. A line is usually located by giving one point on it and its direction or slope, by giving two points on it, or by some other way which is essentially equivalent to one of these. Five typical forms of the equation of a straight line will be discussed. These forms are related and can be transformed from one into the other by the ordinary algebraic operations on equations.

16. Equation of a line through a given point and with a given slope. Let $P_1(x_1, y_1)$ be the given point, and let m be

the slope of a line through P_1 . If $P(x, y)$ is any other point on the line, the slope of the line segment P_1P is m . The slope of the line segment P_1P is $(y - y_1)/(x - x_1)$. Hence the equation of the line through P_1 with slope m is

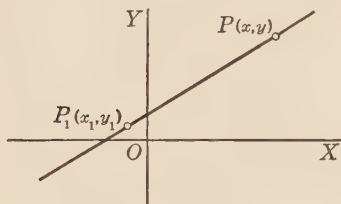


FIG. 15

$$\frac{y - y_1}{x - x_1} = m,$$

$$\text{or } y - y_1 = m(x - x_1). \quad (1)$$

If the point (x_1, y_1) is the origin $(0, 0)$, the equation becomes

$$y = mx.$$

In case the line is parallel to the x -axis, the slope $m = 0$, and the equation becomes

$$y = y_1.$$

In case the line is parallel to the y -axis the angle of inclination is 90° . But $\tan 90^\circ$ is not a definite number and the slope form of the equation cannot be used. A line parallel to the y -axis is at a constant distance from it and hence, for such a line through P_1 ,

$$x = x_1.$$

For example, the equations of the two lines through $(-2, 6)$ parallel to the x - and y -axes, respectively, are $y = 6$ and $x = -2$.

EXAMPLE. Write the equation of the line through $(1, -2)$ perpendicular to the line segment joining $(1, -2)$ and $(4, 3)$.

The slope of the segment joining $(1, -2)$ and $(4, 3)$ is $\frac{5}{3}$. The slope of the required line is therefore $-\frac{3}{5}$. Hence

$$y + 2 = -\frac{3}{5}(x - 1), \quad \text{or} \quad 3x + 5y + 7 = 0,$$

is the equation desired.

17. Equation of a line through two given points. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two given points on a line whose equation is required, and let $P(x, y)$ be any other point on the

line. The slope of the line segment PP_1 must be the same as the slope of the line segment P_1P_2 . Hence the equation of the line is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

or $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \quad (2)$

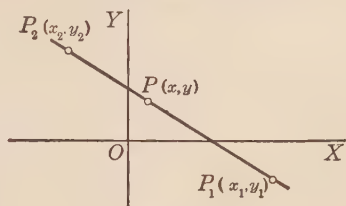


FIG. 16

This is essentially the same as the case of a line through a point and with a given slope, since

$(y_2 - y_1)/(x_2 - x_1)$ is the slope of the line joining (x_1, y_1) and (x_2, y_2) . If $x_1 = x_2$ the line is parallel to the y -axis and the foregoing equation is not valid. This case has been discussed in the preceding section.

EXAMPLE. Write the equation of the line passing through $(-1, -4)$ and $(5, 5)$.

The slope is $\frac{3}{2}$. The equation of the line through the point $(-1, -4)$ and with slope $\frac{3}{2}$ is

$$y + 4 = \frac{3}{2}(x + 1), \quad \text{or} \quad 3x - 2y - 5 = 0.$$

The student should verify that the coördinates of the point $(5, 5)$ satisfy this equation.

18. Definition of intercepts. The intercepts of a line on the axes are the directed distances from the origin to the points where the line crosses the axes. To find the y -intercept let x equal zero in the equation of the line and solve for y ; and to find the x -intercept let y equal zero and solve for x . For example, the x - and y -intercepts of the line $3x - 2y - 5 = 0$ are $\frac{5}{3}$ and $-\frac{5}{2}$, respectively.

EXERCISES

1. Find the intercepts of the following lines and draw each line:

(a) $2x + y = 6.$

(c) $5x + 3y + 10 = 0.$

(b) $4y - 3x + 12 = 0.$

(d) $y = 5x + 5.$

2. Draw the following lines:

(a) $3x - 2y = 0.$

(c) $y = 4.$

(b) $3x + 2y = 0.$

(d) $x = -3.$

3. Which of the following points lie on any of the lines in Exercises 1 and 2?

- (a) (0, 0). (c) (1, 1). (e) (4.5, 1.5).
 (b) (1, 10). (d) (-3, 20). (f) ($\frac{1}{2}$, 4).

4. Write the equation of the line through

- (a) (4, -1) with slope 2.
 (b) (-6, 3) with slope $-\frac{2}{3}$.
 (c) ($\frac{1}{3}$, $-\frac{2}{3}$) with slope $\frac{3}{2}$.
 (d) (5, 8) with slope 0.
 (e) (a, b) with slope b/a .
 (f) $(1 + \sqrt{2}, 1 - \sqrt{2})$ with slope 1.
 (g) (0, 0) with inclination angle 45° .
 (h) (0, 0) with inclination angle 135° .
 (i) (0, 1) with inclination angle 60° .
 (j) (2, 0) with inclination angle 90° .

5. Write the equation of the line determined by the following points:

- (a) (2, 5) and (3, 7). (f) (0, 0) and (2, 5).
 (b) (-4, 9) and (0, 6). (g) (4, 2) and (4, -2).
 (c) (0, 4) and (3, 0). (h) (0, 0) and $(\sqrt{3}, -\sqrt{6})$.
 (d) $(\frac{1}{2}, \frac{2}{3})$ and $(\frac{3}{4}, 0)$. (i) (a, b) and (-a, -b).
 (e) (0, 8) and (1, 8). (j) (a + b, c + d) and (a - b, c - d).

6. Write the equations of the four sides and the two diagonals of the rectangle whose vertices are (0, 0), (6, 0), (6, 4), and (0, 4).

7. Write the equations of the three sides of the triangle (-8, 0), (8, 0), and (0, 8).

8. Write the equations of the three medians of the triangle (0, 0), (6, 8), and (10, 0).

9. An isosceles right triangle lying in the first quadrant has the ends of its hypotenuse at (0, 0) and (0, 12). Write the equations of its three sides.

10. An equilateral triangle lying in the first quadrant has two of its vertices at (0, 0) and (a, 0). Write the equations of its three sides.

11. Show that (-1, 0.3), (0, 1), (3, 3.1), (5, 4.5), and (10, 8) all lie on a straight line.

12. Write the equation of the line through the origin which bisects that portion of the line $8x - 3y + 24 = 0$ which is included between the coördinate axes.

13. Write the equations of the two lines through $(3, -2)$ parallel and perpendicular, respectively, to the line through $(5, 3)$ and $(9, 0)$.

14. Write the equations of the two diagonals of the square two of whose opposite vertices are $(-1, -3)$ and $(5, 7)$.

15. Write the equations of the two diagonals of the rectangle three of whose vertices are $(-2, \frac{5}{2})$, $(3, -\frac{5}{2})$, and $(7, \frac{3}{2})$.

16. A point moves so that the sum of the squares of its distances from each of two adjacent sides of a square is equal to the square of its distance from the point of intersection of the other two sides of the square. Show analytically that its locus is a straight line and determine its position relative to the square.

19. Equation of a line in terms of its slope and y -intercept. A line whose y -intercept is b passes through the point $(0, b)$. Hence the equation of a line with slope m and y -intercept b can be obtained by using equation (1). The equation is, therefore,

$$y - b = m(x - 0), \quad \text{or} \quad y = mx + b.$$

The latter equation puts in evidence the slope m and the y -intercept b .

20. Equation of a line in terms of its intercepts on both axes. If the x -intercept is a and the y -intercept is b , the line passes through the two points $(a, 0)$ and $(0, b)$. Hence its equation is

$$y - b = -\frac{b}{a}(x - 0), \quad \text{or} \quad bx + ay = ab.$$

Dividing by ab , we have

$$\frac{x}{a} + \frac{y}{b} = 1.$$

This equation puts in evidence the intercepts a and b .

21. Theorem. *Every straight line can be represented by a first degree equation in two variables in rectangular coördinates; and conversely, every first degree equation in two variables in rectangular coördinates represents a straight line.*

A line is determined by a point on it and by its direction through the point. If the line is parallel to the y -axis, it can

be represented by an equation of the form $x = a$, which is of the first degree. If the line is not parallel to the y -axis, a point on it and its direction are sufficient to enable us to find its equation by the method of Sec. 16. We have seen that the equation so derived is of the first degree. Hence the first part of the theorem is proved.

The general equation of the first degree in two variables is

$$Ax + By + C = 0,$$

where A , B , and C are constants. If $B \neq 0$ the equation can be written in the form

$$y = mx + b,$$

where $m = -A/B$ and $b = -C/B$. Transposing b and dividing by x , we have

$$\frac{y - b}{x} = m. \quad (3)$$

From the figure we see that equation (3) expresses the fact that the line joining the point (x, y) to the point $(0, b)$ has the constant slope m . This is true when, and only when, the point (x, y) lies on a straight line through $(0, b)$ with slope m .

If $B = 0$ the equation becomes

$$x = k,$$

where $k = -C/A$, and this is true when, and only when, the

point (x, y) lies on a line parallel to the y -axis and distant k units from it. Hence every first degree equation in two variables represents a straight line. Since the equation $Ax + By + C = 0$ represents a straight line, it is called a **linear equation**.

22. Reduction of the general equation to the slope and y -intercept form, and to the intercept form. It is often desirable to reduce an equation of the form $Ax + By + C = 0$ to the form $y = mx + b$, because this latter form readily

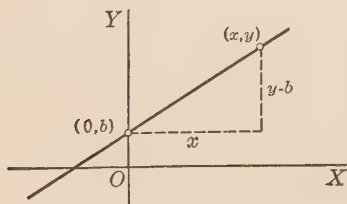


FIG. 17

shows the slope m and the y -intercept b of the line. Thus the equation

$$2x + 5y - 15 = 0$$

when solved for y becomes

$$y = -\frac{2}{5}x + 3.$$

This equation represents a line with slope $-\frac{2}{5}$ and with y -intercept 3.

To reduce an equation of the form $Ax + By + C = 0$ to the intercept form, solve for the intercepts directly from the equation and substitute in the intercept form

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Thus the line whose equation is

$$2x + 5y - 15 = 0$$

has x -intercept $a = \frac{15}{2}$ and y -intercept $b = 3$. The intercept form of the equation is, therefore,

$$\frac{x}{\frac{15}{2}} + \frac{y}{3} = 1.$$

23. Point of intersection of two straight lines. The coördinates of the point of intersection of two straight lines must satisfy the equation of each line. Hence the point of intersection of two straight lines can be found by solving their equations simultaneously.

EXERCISES

1. Reduce each of the following equations to the slope and y -intercept form. Determine the slope and y -intercept and draw each line.

(a) $x + 3y = 1.$

(e) $3y + 6x = 12.$

(b) $5x - y + 8 = 0.$

(f) $4x - y = 0.$

(c) $y - 3x = 2.$

(g) $2x = 6 - y.$

(d) $x + 4y = 0.$

(h) $10x = 2y + 3.$

2. Which lines in Exercise 1 are parallel and which pairs are perpendicular?

3. Reduce equations (a), (b), (c), (e), (g), and (h) in Exercise 1 to the intercept form. Why cannot (d) and (f) be reduced to the intercept form?

4. Write the equations of the lines through $(-3, 5)$ parallel and perpendicular, respectively, to the line $5x + 3y - 2 = 0$.

5. Write the equation of the perpendicular bisector of the line segment joining $(-2, 3)$ and $(6, -9)$.

6. Find the point on the line $6x - 3y - 8 = 0$ equidistant from $(2, -7)$ and $(4, 5)$.

7. The diameter of a circle lies on the line $4x - 3y = 17$ and has for one end the point whose abscissa is 5. Find the coördinates of the other end if the circle passes through $(7, -3)$.

8. Find the length of the perpendicular dropped from $(1, 2)$ upon the line passing through $(0, -5)$ and $(12, -1)$.

9. The midpoints of the sides of a triangle are $(0, 0)$, $(5, -2)$, and $(4, 4)$. Write the equations of the sides of the triangle.

10. How far must each of the nonparallel sides of the trapezoid whose vertices are $(-3, -4)$, $(-1, 2)$, $(8, 5)$, and $(12, 1)$ be extended until they intersect?

11. Write the equations of the sides of the square two of whose opposite vertices are $(-3, 2)$ and $(1, -6)$ and find the other two vertices.

12. Find the angle of intersection of the lines $2y = x$ and $3x + 4y = 40$.

13. Find the angle of intersection of the line $2x - 3y + 1 = 0$ with the y -axis.

14. Find the angles of the triangle formed by the lines $y = 0$, $3y = 4x$, and $3x + 4y = 21$.

15. Find the angles of the triangle formed by the lines $x = 1$, $x + 6y = 19$, and $x - 2y = 3$.

16. Show that the lines $4x - y - 1 = 0$, $3x + 2y - 20 = 0$, and $5x - 3y + 11 = 0$ intersect in a common point.

17. The vertices of a triangle are $(-5, 3)$, $(5, -3)$, and $(7, 3)$. Write the equations of the three medians and show that the medians intersect in a common point.

18. Write the equations of the perpendicular bisectors of the three sides of the triangle in Exercise 17 and show that these lines meet in a common point.

19. Write the equations of the perpendiculars through the vertices upon the opposite sides of the triangle in Exercise 17 and show that these lines meet in a common point.

20. Show that the points of intersection referred to in Exercises 17, 18, and 19 lie on a straight line.

21. Show that the equation $A_1x + B_1y + C_1 = 0$ and the equation $A_2x + B_2y + C_2 = 0$ represent lines that are (a) parallel if $A_1/A_2 = B_1/B_2$; (b) perpendicular if $A_1/A_2 = -B_2/B_1$; (c) coincident if $A_1/A_2 = B_1/B_2 = C_1/C_2$.

22. Derive the equation of a line in terms of its slope and the x -intercept.

23. Show analytically that the locus of points equidistant from (x_1, y_1) and (x_2, y_2) is the perpendicular bisector of the line segment joining the two points.

24. Show that the medians of a triangle whose vertices are (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) meet in the point

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right).$$

25. A point moves so that the sum of the squares of its distances from two vertices of an equilateral triangle is equal to twice the square of its distance from the third vertex. Show analytically that its locus is a straight line parallel to a side of the triangle and passing through the point of intersection of the medians.

24. Normal equation of a line. The normal equation of a line is the equation in terms of the perpendicular distance to the line from the origin and the positive angle which that perpendicular makes with the positive end of the x -axis. The distance measured from the origin to the line will be considered positive. In the figure let $p = OP$ be the perpendicular distance from the origin to the line, and let ω be the angle which OP makes with the positive end of the x -axis. The coördinates of the point of intersection P of OP with the line AB are $(p \cos \omega, p \sin \omega)$.

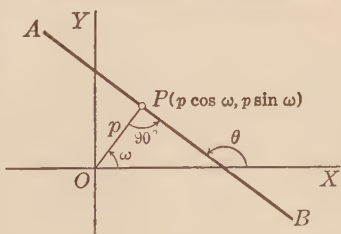


FIG. 18

The inclination angle θ is equal to $90^\circ + \omega$, and hence the slope m may be written

$$m = \tan \theta = \tan (90^\circ + \omega) = -\cot \omega.$$

The equation of the line AB is (see Sec. 16)

$$y - p \sin \omega = -\cot \omega (x - p \cos \omega).$$

Replacing $\cot \omega$ by $\cos \omega / \sin \omega$, we can reduce the equation to

$$x \cos \omega + y \sin \omega = p(\sin^2 \omega + \cos^2 \omega) = p,$$

or

$$x \cos \omega + y \sin \omega - p = 0.$$

This equation is called the **normal equation** of a straight line.

In case the line crosses a quadrant other than the first, a similar argument leads to an equation of the same form.

EXAMPLE. Write the normal equations of the two lines each of which is distant 5 units from the origin and has an inclination angle of 45° .

Since the inclination angle of each line is 45° , the lines are parallel, and for the line AB , ω is 135° , and for the line CD , ω is 315° . The equations are

$$[AB] \quad x \cos 135^\circ + y \sin 135^\circ - 5 = 0$$

$$\text{and } [CD] \quad x \cos 315^\circ + y \sin 315^\circ - 5 = 0,$$

$$\text{or } [AB] \quad -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y - 5 = 0$$

$$\text{and } [CD] \quad \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y - 5 = 0.$$

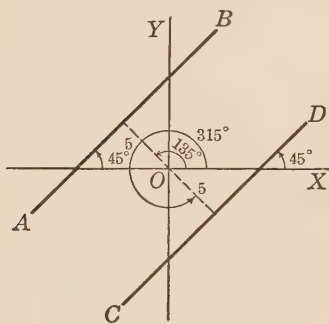


FIG. 19

25. Reduction of the general first degree equation to the normal form. If the general equation of a line,

$$Ax + By + C = 0,$$

and the normal equation,

$$x \cos \omega + y \sin \omega - p = 0,$$

represent the same line, it is possible to multiply one equation by a constant such that the two equations become identical (see Exercise 21 on page 27). Let us multiply the first equation by k , where k is chosen so that

$$kA = \cos \omega, \quad kB = \sin \omega, \quad \text{and} \quad kC = -p.$$

Squaring the first two of these equations and adding, we have

$$k^2(A^2 + B^2) = \cos^2 \omega + \sin^2 \omega = 1.$$

Hence
$$k = \frac{1}{\pm \sqrt{A^2 + B^2}}.$$

Multiplying the general equation by this value of k , we obtain the normal equation

$$\frac{Ax}{\pm \sqrt{A^2 + B^2}} + \frac{By}{\pm \sqrt{A^2 + B^2}} + \frac{C}{\pm \sqrt{A^2 + B^2}} = 0,$$

where the sign before the radical must be chosen opposite to the sign of C in order to agree with the normal form. In case $C = 0$, the sign of the radical is chosen the same as the sign of B , so that $\sin \omega$ is positive, that is, so that ω shall be less than 180° .

EXAMPLE. Reduce the equation $2x - 3y + 6 = 0$ to the normal form.

The constant multiplier which will reduce the equation to the normal form is found to be $1/(-\sqrt{13})$. Hence the required form is

$$-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} - \frac{6}{\sqrt{13}} = 0.$$

From this equation we see that for the given line

$$\cos \omega = -\frac{2}{\sqrt{13}}, \quad \sin \omega = \frac{3}{\sqrt{13}}, \quad \text{and} \quad p = \frac{6}{\sqrt{13}}.$$

26. Distance from a line to a point. The distance from the line AB (Fig. 20) to the point (x_1, y_1) can be found in the following manner. The normal equation of the line AB is

$$x \cos \omega + y \sin \omega - p = 0.$$

If a line CD is drawn through the point (x_1, y_1) parallel to AB , its equation will evidently be

$$x \cos \omega + y \sin \omega - p_1 = 0,$$

where $p_1 = OM$. But $OM = p + d$, where d is the distance from the line AB to the point (x_1, y_1) . Since the point (x_1, y_1) is on the line CD ,

$$x_1 \cos \omega + y_1 \sin \omega - (p + d) = 0,$$

$$\text{or } d = x_1 \cos \omega + y_1 \sin \omega - p.$$

Therefore, the distance from a line to a point can be found by writing the equation of the line in normal form and substituting in the left member of the equa-

tion the coördinates of the point. The result is the distance from the line to the point, and is zero only if the point is on the line. If the point (x_1, y_1) lies on the opposite side of the line AB from the origin, the distance d is positive, and if the point lies on the same side of the line as the origin, the distance d is negative.

EXAMPLE. Find the distance from the line $7x + y - 10 = 0$ to the point $(3, 4)$.

Reducing the equation to the normal form, we have

$$\frac{7x + y - 10}{5\sqrt{2}} = 0.$$

Substituting in the left member the coördinates of the given point, we have

$$d = \frac{7 \cdot 3 + 1 \cdot 4 - 10}{5\sqrt{2}} = \frac{15}{5\sqrt{2}} = \frac{3\sqrt{2}}{2},$$

the required distance. Since the distance is positive, the given point $(3, 4)$ is on the opposite side of the line from the origin.

27. Equation of the bisector of an angle. **EXAMPLE.** Find the equations of the bisectors of the angles formed by the lines $3x + 4y + 10 = 0$ and $5x - 12y - 12 = 0$.

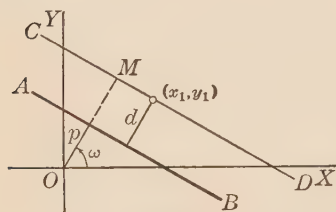


FIG. 20

We make use of the fact that the bisector of an angle is the locus of points equidistant from the sides of the angle. From the figure we have

$FP = EP$ and $MP' = -NP'$, where FP and EP have the same signs and MP' and NP' have opposite signs, since P is on the same side of each line as the origin and P' is not. But

$$FP = \frac{3x' + 4y' + 10}{-5},$$

$$EP = \frac{5x' - 12y' - 12}{13},$$

$$MP' = \frac{3x'' + 4y'' + 10}{-5}, \text{ and } NP' = \frac{5x'' - 12y'' - 12}{13}.$$

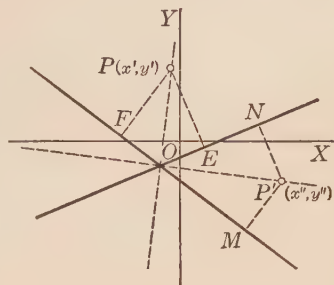


FIG. 21

Hence the locus of P is

$$\frac{3x' + 4y' + 10}{-5} = \frac{5x' - 12y' - 12}{13},$$

and the locus of P' is

$$\frac{3x'' + 4y'' + 10}{-5} = -\frac{5x'' - 12y'' - 12}{13}.$$

Clearing of fractions, reducing, and omitting prime marks, we have as the equations of the two bisectors

$$32x - 4y + 35 = 0 \quad \text{and} \quad 7x + 56y + 95 = 0.$$

It will be observed that these lines have slopes which are negative reciprocals, and the lines are, therefore, perpendicular. This was proved in elementary geometry and gives a check on our work.

EXERCISES

1. Draw the lines for which

(a) $\omega = 30^\circ$, $p = 5$.

(b) $\omega = 135^\circ$, $p = 3$.

(c) $\omega = 180^\circ$, $p = 4$.

(d) $\omega = 300^\circ$, $p = 6$.

(e) $\omega = 2\pi/3$, $p = 0$.

(f) $\omega = 5\pi/4$, $p = 10$.

(g) $\omega = -3\pi/4$, $p = 10$.

(h) $\omega = \pi/2$, $p = 4$.

2. Reduce the following equations to the normal form, determine ω and p , and draw each line:

$$(a) 3x - 4y - 10 = 0.$$

$$(d) x + y = 0.$$

$$(b) x + y = 4.$$

$$(e) x - y = 0.$$

$$(c) x + \sqrt{3}y + 6 = 0.$$

$$(f) y = mx + b.$$

3. Derive the normal form from the intercept form of the equation of the straight line.

4. Find the distance from

$$(a) 8x + 15y - 30 = 0 \text{ to } (1, 6). \quad (c) 3x + 4y = 0 \text{ to } (2, 1).$$

$$(b) 8x + 15y - 30 = 0 \text{ to } (1, 1). \quad (d) 3x - 4y = 0 \text{ to } (2, 1).$$

5. Show that $(2, 6)$ and $(3, 10)$ lie on opposite sides of the line $24x - 7y - 5 = 0$.

6. Show that $(1, 0.6)$ lies between the parallel lines $x + 2y = 2$ and $5x + 10y = 12$.

7. Find the distance between the parallel lines

$$(a) y = -2x + 10 \text{ and } y = -2x + 15.$$

$$(b) x - 3y + 20 = 0 \text{ and } x - 3y - 10 = 0.$$

8. Find the length of each of the three altitudes of the triangle whose vertices are $(2, 5)$, $(2, -5)$, and $(10, 1)$.

9. Find the equations of the loci of points at the distances ± 3 from $12x - 5y - 39 = 0$.

10. Find the equation of the locus of points numerically equidistant from the parallel lines in each set of Exercise 7.

11. Find the points equidistant from $(0, 10)$ and $(8, 6)$ and at the distances ± 4 from the line $4x + 3y - 10 = 0$.

12. Find the area of the triangle whose vertices are $(0, -2)$, $(7, -1)$, and $(5, 3)$.

13. Find the area of the trapezoid formed by the lines $x + 3y = 9$, $x + 3y = 25$, $x + y = 3$, and $3x - 7y = 27$.

14. Find the equations of the bisectors of the angles between the lines $x + 3y - 6 = 0$ and $6x + 2y - 3 = 0$.

15. Find the equations of the bisectors of the angles between the x -axis and the line $4y = 3x$.

16. Find the equations of the bisectors of the angles of the triangle formed by the lines $3x + 4y - 12 = 0$, $4x - 3y + 9 = 0$,

and $8x - 15y - 54 = 0$, and show that these bisectors intersect in a common point.

17. Find the radius of the inscribed circle of the triangle in Exercise 16.

28. Constants in the equation of a straight line. The theorem of Sec. 21 states that the graph of a first degree equation is a straight line. The constants in the equation determine what particular line a first degree equation represents. With few exceptions the equation of a given line may be written in any one of the five forms discussed. In each case the constants give enough properties of the line to determine it. The equation we use depends upon the properties of the line which we know, or upon the properties of the line which we want the equation to exhibit. The equations $2x + y - 5 = 0$ and $y = -2x + 5$ represent the same line. Any property of the line that can be derived from one may also be derived from the other. In the second form, however, the equation shows at once the slope and y -intercept.

The slope and y -intercept, the intercept, and the normal forms of the equation of a straight line each contain two constants. The slope and one point and the two point forms apparently each contain more than two constants. In each case, however, the number of constants can be reduced to two. The general equation

$$Ax + By + C = 0$$

has but two essential constants, since if any one of the constants A , B , or C , say A , is different from zero we may divide by it, thus reducing the equation to

$$x + dy + e = 0,$$

where $d = B/A$ and $e = C/A$. The constants we have used are five in number, the slope m , the intercepts a and b on the axes, the perpendicular distance p from the origin, and the direction angle ω of the perpendicular.

Two geometric conditions determine a line, such as two points, or a point and a direction. Two constants are

sufficient to make an equation satisfy two conditions. For example, if we want the line $y = mx + b$ to pass through the point $(3, 2)$, we have

$$2 = 3m + b,$$

since the coördinates of a point on a line satisfy the equation of the line. *Thus to the geometric condition that the point is on the line corresponds the algebraic condition $2 = 3m + b$, showing a relation between the constants.* A second geometric condition will give a second algebraic equation corresponding to that condition. From our study of simultaneous equations in algebra we recall that two equations are sufficient to determine two unknowns. *Hence we may put as many geometric conditions on a line as there are undetermined constants in the equation of a line, namely, two.*

29. Families of lines. When we consider the equation $y = 2x + b$, we see that it represents a line with slope 2 for each value of b . Giving particular values to b , we have parallel lines with different y -intercepts. Such a system of lines, depending upon an arbitrary constant, is called a **family of lines**. The arbitrary constant is frequently called a **parameter**. Each of the type forms of the equation of the straight line represents a family of lines depending upon the arbitrary constants, or parameters, contained in it. When the parameters are given definite values, a particular member is selected from the family of lines represented by the equation with parameters. A line can be made to satisfy as many conditions as there are parameters in the type equation.

EXAMPLE. From the family of lines represented by the equation $y = mx + b$ select the one passing through $(3, 3)$ and $(-1, -5)$.

Since these points are on the line, the coördinates of each must satisfy the equation, and we have

$$3 = 3m + b \quad \text{and} \quad -5 = -m + b.$$

Solving simultaneously, we obtain $m = 2$ and $b = -3$. Hence $y = 2x - 3$ is the line of the family $y = mx + b$ which passes through the given points.

30. Lines through the point of intersection of two given lines. The coördinates of the point of intersection of the two lines

$$a_1x + b_1y + c_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2 = 0$$

satisfy the equation

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0, \quad (4)$$

for every value of k , since the substitution of the coördinates of the point of intersection makes each expression in parenthesis equal to zero. Equation (4) is of the first degree and, therefore, represents a straight line. Hence equation (4) represents a straight line through the intersection of the first two lines for every value of k . By the proper choice of the parameter k this line can be made to satisfy one more condition.

EXAMPLE. Find the equation of the line through the intersection of the lines $2x + 3y - 9 = 0$ and $x + 2y - 7 = 0$ and with slope $-\frac{1}{3}$.

The equation

$$2x + 3y - 9 + k(x + 2y - 7) = 0$$

represents a family of lines through the intersection of the given lines. The slope of any member of the family of lines in terms of k is

$$m = -\frac{2+k}{3+2k}.$$

Since the slope of the line whose equation is desired is $-\frac{1}{3}$, we let

$$-\frac{2+k}{3+2k} = -\frac{1}{3}.$$

This gives $k = -3$. Substituting the value of k in the equation representing the family of lines and collecting terms, we have the required equation,

$$x + 3y - 12 = 0.$$

EXERCISES

1. Draw several members of each of the following families of lines and determine the common property possessed by each family:

$$(a) y = 3x + b.$$

$$(e) x \cos \frac{\pi}{3} + y \sin \frac{\pi}{3} - p = 0.$$

$$(b) y = mx + 3.$$

$$(f) y = mx.$$

$$(c) x = k.$$

$$(g) y - 2 = m(x - 3).$$

$$(d) x/a + y/3 = 1.$$

$$(h) x \cos \omega + y \sin \omega - 6 = 0.$$

2. Select the member of the family $y = 2x + b$ which passes through (a) (1, 5); (b) (0, 8).

3. Select the member of the family $y = mx + 2$ which (a) is parallel to $2x + y = 0$; (b) is perpendicular to $2x + y = 0$; (c) passes through (1, 1).

4. Write the equation of the family of lines perpendicular to the line through $(-2, 4)$ and $(8, -2)$, and select the member of this family which bisects the segment joining the given points.

5. Write the equation of the family of lines each of which has its x -intercept twice its y -intercept, and select the member of this family which passes through (10, 1).

6. Write the equation of the family of lines tangent to the circle of radius 5 with center at the origin, and select the member of this family tangent to the circle at the point (4, 3).

7. Select the members of the family in Exercise 6 which are perpendicular to the special member found in that exercise.

8. Write the equation of the line through (2, 4) which forms with the coordinate axes a right triangle of area (a) 25 square units; (b) 16 square units; (c) 12 square units.

9. Write the equation of the line through $(-6, 2)$ and $(4, 7)$, using in turn each of the following forms:

$$(a) y = mx + b; (b) x/a + y/b = 1; (c) x \cos \omega + y \sin \omega - p = 0.$$

10. Write the equation of the line through the point of intersection of $x + 5y - 6 = 0$ and $2x - y + 3 = 0$ and through (a) $(-2, 1)$; (b) $(0, 0)$.

11. Write the equation of the line through the point of intersection of $x + 2y - 5 = 0$ and $3x - 5y + 7 = 0$ and (a) parallel to a line with slope $\frac{1}{2}$; (b) perpendicular to the first of the given lines.

12. Write the equation of the line through the point of intersection of $4x + y - 4 = 0$ and $3x - 8y - 10 = 0$ and having (a) the x -intercept equal to 2; (b) the y -intercept equal to -2 .

13. Write the equation of the line distant $\sqrt{5}$ units from the origin and passing through the intersection of $x - 3y + 1 = 0$ and $2x + 5y - 9 = 0$.

14. Write the equation of the line through the intersection of $3x - y + 7 = 0$ and $8x + 5y - 3 = 0$ and through (a) $(-3, -2)$; (b) $(6, -9)$.

31. **Polar equation of a straight line.** In the figure, $OC = p$ is drawn from the pole perpendicular to the line AB whose equation we are seeking and making an angle ω with the polar axis. Let $P(\rho, \theta)$ be any point on the line AB and draw OP . Then $\angle COP = \theta - \omega$, and from the figure we have

$$\rho \cos(\theta - \omega) = p.$$

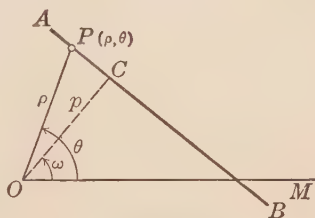


FIG. 22

This is the polar equation of the line AB , since it is a relation between ρ and θ that is true for every point on the line AB and for no other point.

There are some important special cases of the polar equation of the straight line. For a line perpendicular to the polar axis $\omega = 0$ and the equation becomes

$$\rho \cos \theta = p.$$

For a line parallel to the polar axis $\omega = 90^\circ$ and the equation becomes $\rho \cos(\theta - 90^\circ) = p$ or $\rho \sin \theta = p$.

A line through the pole has for its equation

$$\theta = k. \quad (k \text{ a constant})$$

The polar equation may be readily developed from the normal equation. Starting with the normal equation,

$$x \cos \omega + y \sin \omega - p = 0,$$

and transforming to polar coördinates by the substitutions

$$x = \rho \cos \theta, \quad y = \rho \sin \theta,$$

we have $\rho \cos \theta \cos \omega + \rho \sin \theta \sin \omega - p = 0$,

or $\rho (\cos \theta \cos \omega + \sin \theta \sin \omega) = p$,

whence $\rho \cos (\theta - \omega) = p$.

This is the same equation as that already derived from the figure.

EXERCISES

1. A line is perpendicular to the polar axis at a point 5 units distant from the pole. Derive its polar equation directly from a figure.

2. A line is parallel to the polar axis and distant 3 units from it. Derive its polar equation directly from a figure.

3. What family of lines is represented by (a) $\rho = a \sec \theta$?
(b) $\rho = b \csc \theta$?

4. Draw the following lines:

(a) $\rho \cos \theta = 3$.

(d) $\theta = \tan^{-1} \frac{3}{4}$.

(b) $\rho \sin \theta = -1$.

(e) $\rho = 2\sqrt{2} \sec(\theta - \pi/4)$.

(c) $\theta = \pi/3$.

(f) $\rho = 3 \sec(2\pi/3 - \theta)$.

5. Transform the equations of Exercise 4 to rectangular coördinates.

6. Transform the following equations to polar coördinates:

(a) $x = 2$.

(c) $x + 2y = 0$.

(e) $x + y = 1$.

(b) $y = -5$.

(d) $x - 2y = 0$.

(f) $Ax + By + C = 0$.

7. The rectangular coördinates of the vertices of a square are $(0, 0)$, $(4, 0)$, $(0, 4)$, and $(4, 4)$. Write in polar coördinates the equations of its four sides and its two diagonals (a) if the pole is at $(0, 0)$ and the polar axis passes through $(4, 0)$; (b) if the pole is at $(2, 2)$ and the polar axis passes through $(4, 2)$.

8. The vertices of a rectangle in rectangular coördinates are $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b) . Write in polar coördinates the equations of its four sides and its two diagonals if the pole is at $(0, 0)$ and the polar axis passes through $(a, 0)$.

9. Show that $(-2\sqrt{2}, 90^\circ)$, $(2\sqrt{2}, 0^\circ)$, and $(4, 15^\circ)$ lie on a straight line.

10. A regular hexagon of side a has its center at the pole and a vertex on the polar axis. Write the polar equations of its sides.

32. The linear function. Any expression which depends for its value upon x is called a **function of x** . The function $mx + b$ is a general function of the first degree in one variable. If we plot points whose abscissas are values of x and whose ordinates are the corresponding values of $mx + b$, these points will be found to lie on a straight line. This is easily shown by setting $y = mx + b$ and applying the theorem of Sec. 21. Hence $mx + b$ is called a **linear function of x** . The linear function occurs frequently in the study of science and its applications.

EXAMPLE. The Fahrenheit thermometer reading is a linear function of the corresponding centigrade thermometer reading.

If we call C the reading of the centigrade thermometer, the corresponding reading of the Fahrenheit thermometer is

$$\frac{9}{5}C + 32.$$

The Fahrenheit thermometer reading is, therefore, a linear function of the centigrade thermometer reading.

Plotting values of C as abscissas and values of the function $\frac{9}{5}C + 32$ as ordinates, we have the line AB , which shows the relation between Fahrenheit and centigrade thermometer readings.

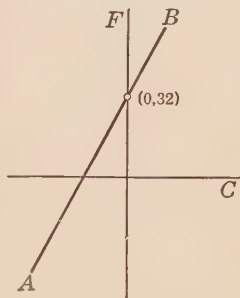


FIG. 23

MISCELLANEOUS EXERCISES

1. Find the intercepts on the axes of the line through $(-2, 3)$ and $(\frac{3}{2}, -\frac{2}{3})$.

2. Write the equations of the families of lines parallel and perpendicular, respectively, to $3x - 7y = 0$.

3. Find the foot of the perpendicular drawn from the origin to the line $2x - 3y + 26 = 0$.

4. Find the point on the line $7x + y = 29$ equidistant from $(-5, 2)$ and $(3, -8)$.

5. Determine c so that the lines $x - y + 2 = 0$, $2x - 3y + 7 = 0$, and $3x + 2y + c = 0$ shall meet in a point.

6. Write the equations of the two lines through the origin which trisect that segment of the line $3x + 5y = 27$ that is included between the coördinate axes.

7. Find the point equidistant from $(-7, 0)$, $(1, -2)$, and $(9, 4)$.

8. The equations of the sides of a triangle are $3x - 4y + 9 = 0$, $13x - 9y = 11$, and $x + 7y + 3 = 0$. (a) Find the lengths of its sides. (b) Find the angles correct to the nearest minute. (c) Find its area.

9. An equilateral triangle lying in the first quadrant has two of its vertices at $(a, 0)$ and $(0, a)$. Write the equations of its three sides.

10. The medians of any triangle meet in a point (see Exercise 24 of Sec. 23). Prove that a similar property holds for (a) the perpendicular bisectors of the sides; (b) the perpendiculars through the vertices upon the opposite sides. (Denote the vertices of the triangle by $(0, 0)$, $(a, 0)$, and (b, c) .)

11. Show that the point of intersection of the medians, that of the perpendicular bisectors of the sides, and that of the perpendiculars through the vertices upon the opposite sides of any triangle lie on a straight line. (Use the same coördinates for the vertices as in Exercise 10.)

12. The ends of the base of a triangle are $(3, -1)$ and $(17, 1)$. Find the locus of the vertex if the area is constant and equal to 30 square units.

13. Find the area of the triangle whose sides are (1) the line $3x - 4y - 17 = 0$, (2) a line through $(5, 12)$ such that the angle between it and (1) is 45° , (3) the line through $(5, 12)$ with slope $-\frac{1}{2}$. (Two solutions.)

14. Find the area of the quadrilateral whose vertices are $(-1, 1)$, $(1, 7)$, $(6, 0)$, and $(5, -2)$.

15. Perpendiculars are dropped from the point $(3, 0)$ upon the sides of the triangle formed by the lines $4x - y + 5 = 0$, $x - 3y + 7 = 0$, and $2x + 5y - 35 = 0$. Show that the feet of these perpendiculars lie on a straight line.

16. The hypotenuse of an isosceles right triangle lies on the line $3x - 5y + 10 = 0$. The vertex is the origin. Find the equations of its two legs.

17. Write the equations of the sides of the square which has for a diagonal the segment of the line $4x - y + 12 = 0$ that is included between the coördinate axes.

18. Two opposite sides and one diagonal of a parallelogram are $3x - 8y + 4 = 0$, $3x - 8y - 24 = 0$, and $x + 2y + 6 = 0$ respectively. The other diagonal passes through $(1, 0)$. Find the four vertices.

19. Find the equations of the bisectors of the two obtuse angles of the parallelogram whose sides are $3x + 4y - 11 = 0$, $3x + 4y + 17 = 0$, $5x - 12y + 19 = 0$, and $5x - 12y - 93 = 0$, and show that these bisectors are parallel.

20. The equations of the two legs of an isosceles triangle are $2y = 7$ and $8x + 6y = 45$. The midpoint of the base is $(1, -\frac{1}{2})$. Find the area.

21. The equations of the two legs of an isosceles triangle are $7x - y + 3 = 0$ and $x + y - 3 = 0$. Find the length of the base if it passes through $(1, -10)$.

22. Write the equation of the line through $(\frac{5}{2}, 5)$ which cuts off equal intercepts on the axes.

23. The point $(5, -3)$ bisects that portion of a line which is included between the coördinate axes. Write the equation of the line.

24. Write the equations of the two lines through $(14, -3)$ each of which has the sum of its intercepts on the axes equal to 10.

25. Write the equations of the two lines through $(4, 3)$ each of which has the product of its intercepts on the axes equal to 75.

26. Write the equations of the lines through $(2, 6)$ at a distance 6 units from the origin.

27. A line is drawn across the first quadrant at a distance $4\sqrt{5}$ units from the origin so that the area of the triangle bounded by it and the axes is 100 square units. Find the equation of the line. (Two solutions.)

28. Find the equations of the two lines intersecting at $(3, 0)$ so that the point $(18, 5)$ is 9 units distant from each line.

29. Show analytically that the two lines drawn from a vertex of a parallelogram to the midpoints of the opposite sides trisect a diagonal of the parallelogram.

30. A point moves so that the slope of the line joining it to $(a, 0)$ equals n times the slope of the line joining it to the origin. Discuss the position of its locus for $n = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.

31. Write the equation of the line passing through the intersection of $x + y + 3 = 0$ and $3x + y + 5 = 0$ and through the intersection of $x - y - 4 = 0$ and $2x - y - 9 = 0$ without finding the point of intersection of either pair of lines.

32. A pole 50 ft. long leans away from a vertical pole 60 ft. long. The bases of the poles are 30 ft. apart, and the leaning pole makes an angle $\tan^{-1} \frac{24}{7}$ with the ground (horizontal). Find the distance from the base of the vertical pole to the point where a line through the tops of the poles strikes the ground.

33. Graph the function $3 - 0.2x$. For what values of x is this function positive? negative?

34. Draw an accurate graph showing the relation between Fahrenheit and centigrade readings on a thermometer (see Sec. 32), and give the Fahrenheit readings on the graph corresponding to the following centigrade readings: $-5^\circ, 0^\circ, 20^\circ, 32^\circ$. Give the centigrade readings on the graph corresponding to the following Fahrenheit readings: $-12^\circ, 0^\circ, 15^\circ, 32^\circ$.

35. If a body is thrown upward with an initial velocity v_0 , its velocity at any subsequent time t is given by the equation $v = v_0 - gt$, where g is approximately 32 when v is in feet and t in seconds. Graph this equation for $v_0 = 160$ feet per second. For what values of t is v positive? negative?

36. A sum of money P at simple interest will in n years amount to $A = P + Prn$, where r is the interest rate. Graph this equation for (a) $P = \$100$ and $r = .06$; (b) $P = \$90$ and $r = .08$; (c) $P = \$75$ and $r = .08$. If (a) and (b) are graphed on the same axes explain the meaning of the intersection of the graphs.

CHAPTER III

EQUATION AND LOCUS

33. Graph of an equation. In Chapter II we have seen that every equation of the first degree in two variables has a straight line associated with it, and conversely. We shall now consider equations of higher degree than the first and their corresponding geometric loci. Consider, for example, the equation $4y = x^3$. It is evidently satisfied by infinitely many pairs of values of x and y . Writing the equation in the form $y = x^3/4$ and assigning values to x , we can easily obtain corresponding values for y . A few of these pairs of values are

$x =$	-4	-3	-2	-1	0	1	2	3	4
$y =$	-16	$-6\frac{3}{4}$	-2	$-\frac{1}{4}$	0	$\frac{1}{4}$	2	$6\frac{3}{4}$	16

Using the pairs of values of x and y found above, we can plot the points shown in Fig. 24. We observe that these points do not lie on a straight line but appear to lie on a smooth curve, as shown in the figure. If additional pairs of values of x and y are found and the corresponding points plotted, a more accurate curve can be drawn. It should be observed that the curve recedes indefinitely far from the axes in the first and third quadrants. The curve recedes more rapidly from the x -axis than from the y -axis since the ordinate y varies as the cube of the abscissa x .

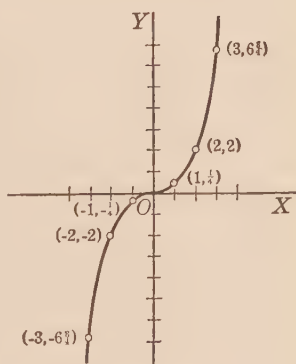


FIG. 24

Definition. The locus or graph of an equation in two variables is the curve* containing all the points, and only the points, whose coordinates satisfy the equation.

Assuming that the curve shown in Fig. 24 satisfies the conditions of the definition, we call it the graph of the equation $4y = x^3$.

As a second illustration consider the equation $xy = 1$. Writing the equation in the form $y = 1/x$ and assigning values to x , we find corresponding values of y and tabulate them as follows:

$x =$	1	2	3	4	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	-1	-2	-3	-4	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$
$y =$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	2	3	4	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	-2	-3	-4

Having plotted these points, the student may be puzzled as to how to join them. As in the preceding example the points are confined to the first and third quadrants, but it should be noted that the value $x = 0$ must be excluded, since for the

value $x = 0$ there is no corresponding value of y . The curve, therefore, does not cut the y -axis, and hence is composed of two separate branches, as shown in the figure.

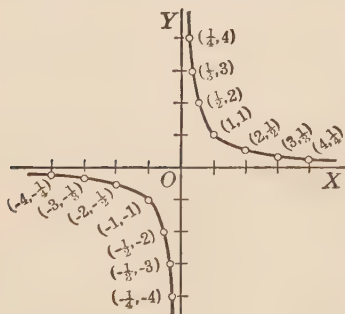


FIG. 25

Although $x = 0$ is excluded, we observe that as x approaches zero through positive values, y increases indefinitely and the branch in the first quadrant recedes indefinitely from the

x -axis and approaches the upward extension of the y -axis. Also as x increases indefinitely through positive values, y decreases and approaches zero. The graph, therefore, recedes indefinitely from the y -axis and approaches the positive ex-

* The word "curve" used in a general sense includes the straight line. The definition of graph in Chapter II is a special case of the general definition here.

tension of the x -axis. A similar discussion shows that the branch in the third quadrant extends indefinitely toward the negative extensions of the x - and y -axes.

EXERCISES

Draw the graphs of the following equations :

1. (a) $2y = x^2$; (b) $2y = x^2 + 2$.
2. (a) $y = x^2 - 3x$; (b) $y = 3x - x^2$.
3. (a) $y = x^3$; (b) $4x = y^3$.
4. (a) $y = x^4$; (b) $3y = x^4$.
5. $y^2 - 3y - x + 2 = 0$.
6. $x^2 - 6x - 3y + 9 = 0$.
7. (a) $y^2 = 3x$; (b) $y^2 = -3x$.
8. (a) $y^2 = x^3$; (b) $x^2 = y^3$.
9. (a) $xy = 9$; (b) $xy = -9$.
10. (a) $x^2y = 10$; (b) $xy^2 = 10$.
11. (a) $x^2 + y^2 = 25$; (b) $x^2 - y^2 = 25$.
12. $y = x^3 + x$.
13. $y = 4x^3 - 3x^2 + 1$.
14. What curves in Exercises 1-13 go through the origin?

34. The two fundamental problems of a first course in analytic geometry. In the preceding chapter we studied two types of problems: (1) given the equation of a line to locate the corresponding line; (2) given the line to find the corresponding equation. We are now interested in considering the similar problems for equations and loci in general:

1. *Given an equation to find the corresponding locus or graph.*
2. *Given a locus to find the corresponding equation.*

This second problem frequently takes the form of finding an equation whose graph approximately fits a set of points given by some sort of observed data. This phase of the second problem is considered in Chapter X.

THE FIRST FUNDAMENTAL PROBLEM

35. Remarks. In Sec. 33 we studied the graphing of an equation by plotting points whose coördinates satisfied the equation. A curve representing an equation can be drawn fairly accurately if a sufficient number of points are plotted. In many cases, however, so many points are required to furnish an accurate conception of the curve that the point method becomes very laborious. Furthermore, the point method in itself furnishes little or no information concerning some of the important properties of the curve. Having plotted many points, the student is often at a loss to know in what order to join them. There are certain properties, however, easily detected from the equation, which if discovered and used will enable one to draw the curve with a minimum amount of point plotting. Some of these properties are discussed in the sections following.

36. Intercepts. The intercepts of a curve are the directed distances from the origin to the points where the curve crosses or touches the axes. To find the x -intercept, let $y = 0$ in the equation of the curve and solve for x . Similarly, to find the y -intercept, let $x = 0$ and solve for y .

37. Symmetry. Two points are symmetric with respect to a line if that line is the perpendicular bisector of the line joining the two points. Two points are symmetric with respect to a third point if that third point is the midpoint of the line joining the first two points. It follows immediately from these definitions that the

point (x, y) is symmetric to

{	$(x, -y)$ with respect to the
	x -axis.
	$(-x, y)$ with respect to the
	y -axis.
	$(-x, -y)$ with respect to
	the origin

A curve is symmetric with respect to a line or with respect to a point if the symmetric point of each point on the curve is also a point on the curve. The curve will be symmetric

with respect to the x -axis if for each point (x, y) on the curve the symmetric point $(x, -y)$ is also on the curve. For example, this is true of the curve whose equation is $x + y^2 = 3$. The equation must be such that the substitution of $-y$ for y does not change its form. This will be true if y occurs in the equation to even powers only.

Similarly, a curve is symmetric with respect to the y -axis if x occurs to even powers only.

A curve is symmetric with respect to the origin if the substitution of $-x$ for x and of $-y$ for y does not change the form of the equation of the curve. Thus the curves of the illustrative examples in Sec. 33 are symmetric with respect to the origin. Symmetry with respect to both axes implies symmetry with respect to the origin, but not conversely.

38. Examples of graphing. EXAMPLE 1. Draw the curve $y^2 = 4x + 4$.

Since y occurs to an even power only there is symmetry with respect to the x -axis. There is no symmetry with respect to the y -axis or the origin.

The intercepts are ± 2 on the y -axis and -1 on the x -axis. Solving the equation for y , we have

$$y = \pm 2\sqrt{x+1},$$

which shows that values of x less than -1 must be excluded, since corresponding values of y would be imaginary. Also, it is seen that y increases numerically as x increases. Hence the curve will recede from the

x -axis as x increases. For $x = -\frac{1}{2}$, $y = \pm 1.4 +$; for $x = 1$, $y = \pm 2.8 +$; for $x = 2$, $y = \pm 3.5 +$; for $x = 3$, $y = \pm 4$; etc. Marking off the intercepts and plotting the points with positive ordinates, we draw a smooth curve ABC (Fig. 26) through them in the order of increasing abscissas. The lower

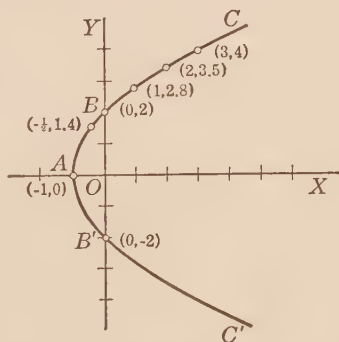


FIG. 26

branch $AB'C'$ is next drawn symmetric to the branch ABC with respect to the x -axis.

It should be noticed that the double sign before the radical in the equation $y = \pm 2\sqrt{x+1}$ also indicates symmetry with respect to the x -axis, since for every value of x there are two values of y numerically equal but opposite in sign.

EXAMPLE 2. Draw the curve $4x^2 + 9y^2 = 36$.

The curve is symmetric with respect to each axis and therefore with respect to the origin. The x -intercepts are ± 3 and the y -intercepts are ± 2 . Solving the equation for y , we have

$$y = \pm \frac{2}{3}\sqrt{9 - x^2}.$$

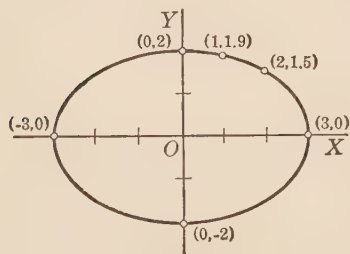


FIG. 27

This equation shows that x cannot be numerically greater than 3. It also shows that for $x = 0$, y has its largest numerical value, namely 2, and that as x increases toward 3, y decreases

numerically toward zero. For $x = 1$, $y = \pm 1.9$; and for $x = 2$, $y = \pm 1.5$. Marking off the intercepts and plotting the points which lie in the first quadrant, we draw the entire curve (Fig. 27), keeping in mind the facts of symmetry and the limitations on x and y .

EXAMPLE 3. Graph the curve $y = x^3 - 4x$.

If we replace x by $-x$ and y by $-y$ the equation becomes

$$-y = -x^3 + 4x \quad \text{or} \quad y = x^3 - 4x.$$

Hence the curve is symmetric with respect to the origin. The x -intercepts are -2 , 0 , and 2 . The y -intercept is 0 . Writing the equation in the form

$$y = (x+2)x(x-2),$$

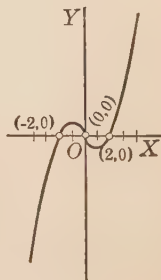


FIG. 28

we see that for $x < -2$, y is negative since each factor in the right member of the equation is negative; for $-2 < x < 0$,

y is positive; for $0 < x < 2$, y is again negative; and for $x > 2$, y is positive. With the information in regard to the change in sign of y , the intercepts, and symmetry with respect to the origin, a rough sketch of the curve can be indicated as in Fig. 28. However, if a few additional points are plotted, such as $(1, -3)$ and $(3, 15)$, a more accurate sketch is obtained (Fig. 29). And even this sketch can be further refined as more points are plotted. We are not sure, for instance, that $(1, -3)$ is the lowest point on that part of the curve, as Fig. 29 seems to indicate.

EXAMPLE 4. Graph the curve

$$x^2 + y^2 - 8x = 0.$$

Solving the equation for x , we have

$$x = 4 \pm \sqrt{16 - y^2}.$$

This equation shows that the curve is symmetric with respect to the line $x = 4$, since for each value of y there are two values of x , one as much less than 4 as the other is greater than 4. The curve is also symmetric with respect to the x -axis. The curve goes through the origin and crosses the x -axis again at $x = 8$. The values of y cannot be greater than 4 nor less than -4 , and the corresponding values of x lie in the interval from 0 to 8. The curve is shown in Fig. 30.

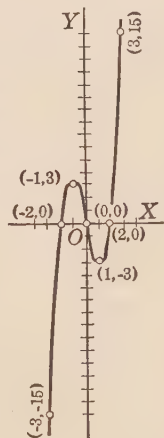


FIG. 29

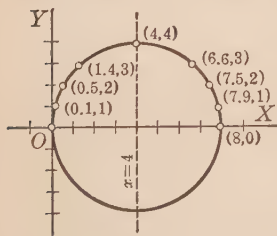


FIG. 30

EXERCISES

Discuss the following equations and draw their graphs:

1. $5y = x^2 + 1.$
2. $y^2 = 9 - 6x.$
3. $x^2 + y^2 = 20.$
4. $2x^2 - 5x - y + 2 = 0.$
5. $x^2 + 4y^2 = 16.$
6. $x^2 - 4y^2 = 16.$

- | | |
|-------------------------------|----------------------------|
| 7. $4y^2 - x^2 = 16$. | 13. $y = x(x+1)(x-2)$. |
| 8. $x^2 + y^2 - 10x = 0$. | 14. $y = x(x+1)(x-2)^2$. |
| 9. $2x^2 + y^2 = 6$. | 15. $y = (x^2 - 4)^2$. |
| 10. $x^2 + y^2 + 4y = 0$. | 16. $y^2 = x^2 - x^4$. |
| 11. $4 + x + 7y - 2y^2 = 0$. | 17. $y^2 = x^4 - x^2$. |
| 12. $y = x^3 - 9x$. | 18. $x^2 + xy + y^2 = 3$. |

39. Asymptotes parallel to an axis. EXAMPLE. Draw the graph of $xy + y = x$.

The curve goes through the origin. There is no symmetry with respect to the axes or the origin. Solving the equation for y , we have

$$y = \frac{x}{x+1}.$$

For $x < -1$, y is positive since both numerator and denominator are negative; for $-1 < x < 0$, y is negative;

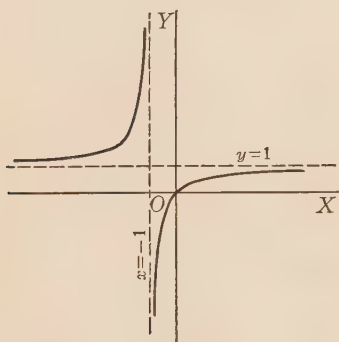


FIG. 31

for x positive, y is positive. As x approaches -1 the numerator of the fraction approaches -1 while the denominator approaches zero. Hence the value of the fraction increases numerically without limit, that is, approaches infinity (∞). As x approaches -1 from the left, y approaches $+\infty$, and as x approaches -1 from the right, y approaches $-\infty$. As x approaches $+\infty$, y approaches 1

but is always less than 1 since the denominator is greater than the numerator. As x approaches $-\infty$, y approaches 1 but is always greater than 1 since the numerator is numerically greater than the denominator. The fact that the curve recedes indefinitely, and approaches the lines $x = -1$ and $y = 1$, aids in drawing the curve (Fig. 31).

Such a line as $x = -1$, or $y = 1$, in the preceding example, which the curve approaches as one of the variables approaches

infinity, is called an **asymptote** of the curve. In some cases the coördinate axes are asymptotes, as in Fig. 25.

40. Graphing by factoring. If the first member of an equation $f(x, y) = 0$ can be factored into two or more factors, as $f(x, y) = f_1(x, y) \cdot f_2(x, y) = 0$, it is evident that only those points (x, y) whose coördinates satisfy either $f_1(x, y) = 0$ or $f_2(x, y) = 0$ will lie on the locus of $f(x, y) = 0$. Hence the graph of $f(x, y) = 0$ will be made up of the graphs of $f_1(x, y) = 0$ and $f_2(x, y) = 0$.

EXAMPLE. Graph

$$x^2y - xy^2 - x + y = 0.$$

Factoring, we have

$$\begin{aligned} x^2y - xy^2 - x + y \\ = (xy - 1)(x - y) = 0. \end{aligned}$$

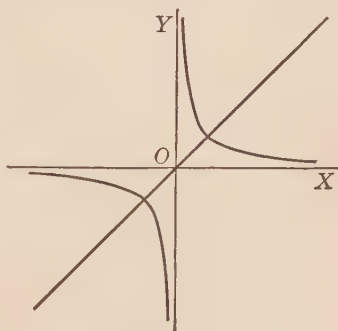


FIG. 32

Hence the required graph is made up of the graphs of the equations $xy - 1 = 0$ and $x - y = 0$. The graph of $xy - 1 = 0$ is shown in Fig. 25, and the graph of $x - y = 0$ is a line through the origin with slope 1. The complete graph is shown in Fig. 32.

41. Summary. Experience will soon teach the student that no fixed rule for finding the graph of an equation will suffice. There are innumerable curves each with different properties, and methods which are effective in one case may not be effective in another. Proficiency in graphing can be attained only by drawing many curves and profiting by the experience gained. The student should, however, ask himself and answer the following questions whenever he has an equation whose graph is required:

- (a) *Does the curve go through the origin?*
- (b) *What are the intercepts on the axes?*
- (c) *Is the curve symmetric with respect to one or both axes or the origin?*

(d) *Is the curve symmetric with respect to a line parallel to an axis?*

(e) *Can the equation be graphed by factoring?*

(f) *Are there any limitations on the values of the variables that satisfy the equation?*

(g) *Do any finite values of one variable make the other variable infinite?*

(h) *What are the equations of the asymptotes parallel to the axes if there are such asymptotes?*

(i) *Do both variables become infinite at the same time? If so, in which quadrant or quadrants?*

EXERCISES

Discuss the following equations and draw their graphs:

$$1. y = \frac{2}{x-1}.$$

$$6. y(x^2 + 1) = 4x. \quad 11. y = \frac{x}{x^3 - 8}.$$

$$2. x = \frac{3}{2+y}.$$

$$7. y(x^2 - 4) = 4. \quad 12. y = \frac{1}{x-1} + \frac{1}{x-5}.$$

$$3. y = \frac{2x}{x+1}.$$

$$8. y(x^2 - 9) = 3x. \quad 13. y = \frac{1}{x-1} - \frac{1}{x-5}.$$

$$4. xy - 3x = 2y.$$

$$9. y(x^2 + 2) = 2x^2. \quad 14. y = \frac{x-1}{x-5}.$$

$$5. y(x^2 + 1) = 4. \quad 10. y = \frac{2}{x^3 - 1}.$$

$$15. y^2 = \frac{x^3}{6-x}.$$

Draw the graphs of the following equations after factoring the left members:

$$16. x^2 - y^2 = 0.$$

$$20. xy + 3x - 2y - 6 = 0.$$

$$17. x^2 + 4xy + 4y^2 - 1 = 0.$$

$$21. 2x^2 + 5x - 6 = 0.$$

$$18. x^2 - 4y^2 + 4y - 1 = 0.$$

$$22. 3x^2 - 8xy - 3y^2 = 0.$$

$$19. x^2y^2 - 2xy = 0.$$

$$23. x^4 - x^3y - xy + y^2 = 0.$$

42. Equation in two variables. Curve. In any equation we have graphed by plotting points we have seen that the points tend to lie on a curve rather than to cover an area. It can be proved that any equation in rectangular coördinates,

$f(x, y) = 0$, represents a curve, isolated points, or no real locus. The proof is beyond the scope of this book, but a short discussion will indicate the manner of proof.

The graph of an equation, $f(x, y) = 0$, has the property that it can be intersected by a straight line in separated points only, that is, in points which do not form a continuous line segment, unless the line is a part of the graph. A plane surface has the property that it may be intersected by a line in an infinite number of points which form a continuous line segment. If we solve $f(x, y) = 0$ and $y = b$ simultaneously, we have the points in which any line parallel to the x -axis cuts the graph of $f(x, y) = 0$. If the points of intersection do not form a continuous line segment for any b , the equation $f(x, y) = 0$ represents a curve or isolated points. Eliminating y between the two equations, we have $f(x, b) = 0$. This is an equation in the single variable x , roots of which are the abscissas of the points of intersection of the line and the curve. The roots of this equation, in the case of the functions which usually arise, are proved in more advanced mathematics to be separated so that the points of intersection of $y = b$ and $f(x, y) = 0$ cannot form a continuous line segment unless the line $y = b$ is a part of the graph of $f(x, y) = 0$. Hence the equation $f(x, y) = 0$ represents a curve or isolated points if it represents a real locus.

Some equations are satisfied by the coördinates of only one point, or of a finite number of points, and their graphs are called **point loci**. The equation $x^2 + y^2 = 0$ is satisfied by the coördinates of the point $(0, 0)$ only. The equation $(x^2 - 1)^2 + (y^2 - 1)^2 = 0$ is satisfied by the coördinates of the four points $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$, and of no other points.

In what we have said it is understood that we mean points with coördinates which are real numbers. There are equations, such as $x^2 + y^2 = -1$, which are satisfied by no pair of real numbers x and y . Hence such equations have no graphical representation in our system of coördinates and are said to represent **imaginary loci**.

43. Graphs of polar equations. EXAMPLE 1. Draw the graph of $\rho = 2 \sin \theta$.

Assigning values* to θ and finding corresponding values of ρ , we make a table of values:

$\theta =$	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\rho =$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

Plotting the corresponding points and drawing a smooth curve through them in the order of increasing θ , we have the graph shown in Fig. 33. It is

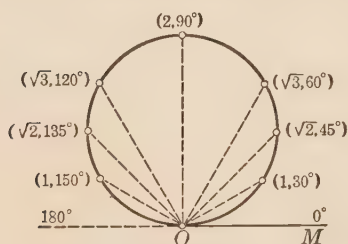


FIG. 33

to be noted that if θ is allowed to vary from 180° to 360° , the corresponding values of ρ will be numerically equal to those obtained above but will be negative. Hence these additional points will coincide with those already found and no new part of the curve will be

obtained. The graph is, therefore, completed as θ varies from 0° to 180° .

EXAMPLE 2. Draw the graph of $\rho = 4(1 + \cos \theta)$.

Making a table of values,

$\theta =$	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\rho =$	8	7.5	6.8	6	4	2	1.2	0.5	0

plotting the corresponding points, and drawing a smooth curve through them in the order of increasing θ , we obtain the branch MBO of the curve in Fig. 34. If θ is allowed to vary from 180° to 360° the corresponding values of ρ will be equal, in reverse order, to those obtained above and will be positive. Hence additional points are found and the branch $OB'M$ is drawn.

* The student should not overlook those values of θ for which the trigonometric function under consideration has maximum, zero, or minimum values.

EXAMPLE 3. Draw the graph of $\rho = a\theta$.

Assigning positive values (in radian measure) to θ and finding the corresponding values of ρ , we plot points and draw the curve as shown in Fig. 35 for $a = 1$. Since ρ varies directly as θ , ρ will

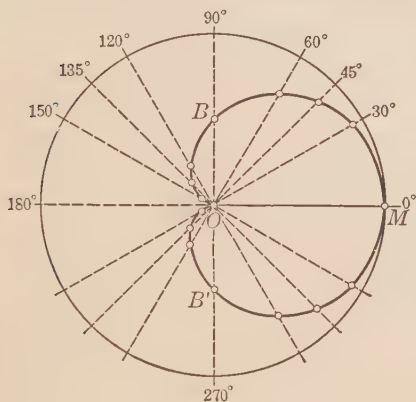


FIG. 34

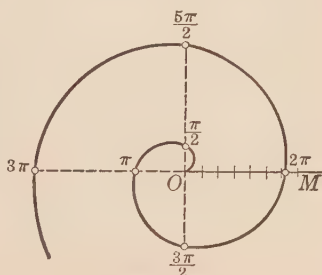


FIG. 35

increase indefinitely as θ increases. Hence the curve will recede in spiral form farther and farther from the pole.

The student should draw that part of the graph corresponding to negative values of θ .

EXERCISES

Draw the graphs of the following equations:

1. $\rho = 3 \cos \theta$. 6. $\rho = 2(1 + \sin \theta)$. 11. $\rho = 5$.
2. $\rho \cos \theta = 3$. 7. $\rho = 2(1 - \sin \theta)$. 12. $\rho = 4(\cos \theta + \sin \theta)$.
3. $\rho = 6 \sin \theta$. 8. $\rho = 1 + 2 \cos \theta$. 13. $2\rho = \theta$.
4. $\rho = 2 \sin^2 \theta$. 9. $\rho = 2 + \cos \theta$. 14. $\rho = \tan \theta$.
5. $\rho = 4\sqrt{\cos \theta}$. 10. $\rho(2 - \cos \theta) = 1$. 15. $\rho = 4 \sec \theta \tan \theta$.

Transform to rectangular coördinates:

16. $\rho = 2 \tan^2 \theta \sec \theta$. 20. $\rho = 10 \cos \theta$.
17. $\rho^2 = 4 \tan \theta \sec^2 \theta$. 21. $\rho = \frac{1}{1 - \cos \theta}$.
18. $\rho^2 = 2 \csc 2\theta$. 22. $\rho = \frac{2(\cos \theta + \sin \theta)}{\sin 2\theta}$.
19. $\rho = a$.

Transform to polar coördinates:

23. $x^2 + y^2 = 10x$.

25. $x^2 - y^2 = 10x$.

24. $x^2 + y^2 + 10y = 0$.

26. $y^2 = x^2 - x^4$.

44. Intersections of curves. If two curves are drawn on the same set of axes it is evident that the coördinates of the points of intersection of the two curves must satisfy the equation of each curve. Hence, to find the coördinates of the points of intersection of two curves, solve their equations simultaneously.

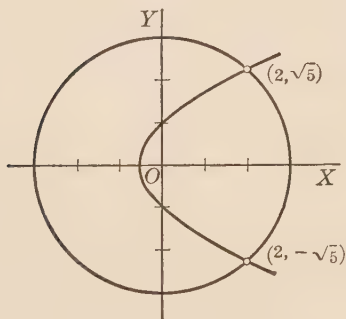


FIG. 36

EXAMPLE 1. Find the points of intersection of the curves $x^2 + y^2 = 9$ and $y^2 = 2x + 1$.

Solving the two equations simultaneously, we have

$$x = 2, y = \pm\sqrt{5} \quad \text{and} \quad x = -4, y = \pm\sqrt{-7}.$$

Thus there are four pairs of algebraic solutions, only two of which are real. Hence the curves intersect in but two real points, $(2, \sqrt{5})$ and $(2, -\sqrt{5})$ (Fig. 36).

EXAMPLE 2. Find the points of intersection of $y^2 = x^3$ and $3x - y = 4$.

Eliminating y from the two equations, we have the equation

$$x^3 - 9x^2 + 24x - 16 = 0,$$

whose roots are found to be 1, 4, 4. Hence there are three pairs of solutions, $(1, -1)$ and $(4, 8)$ twice. The significance of the repeated solution $(4, 8)$ is shown in Fig. 37. At this point the line is tangent to the curve.

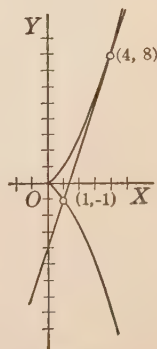


FIG. 37

Two equations when treated simultaneously may have no solution or may have imaginary solutions only. In either case the graphs of the equations do not intersect.

EXAMPLE 3. Find the points of intersection of $xy = 9$ and $x + y = 1$.

Solving for x , we find

$$x = \frac{1 \pm \sqrt{-35}}{2}.$$

Hence the two loci do not intersect in real points (Fig. 38).

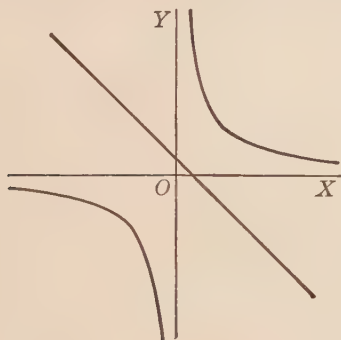


FIG. 38

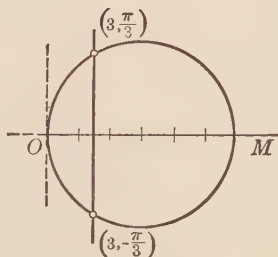


FIG. 39

EXAMPLE 4. Find the points of intersection of $\rho = \frac{3 \sec \theta}{2}$ and $\rho = 6 \cos \theta$.

Solving simultaneously, we have

$$\cos \theta = \pm \frac{1}{2}.$$

Whence we find that the points of intersection are $(3, \pi/3)$ and $(3, -\pi/3)$ (Fig. 39).

EXERCISES

Find the points of intersection of the graphs of the following equations and check the results graphically:

1. $y^2 = 9x$ and $3x - 2y - 9 = 0$.
2. $y^2 = 9x$ and $3x - 2y + 3 = 0$.
3. $y^2 = 9x$ and $3x - 2y + 4 = 0$.
4. $x^2 + y^2 = 25$ and $3x + 4y = 0$.
5. $2y^2 = 9x$ and $3x^2 = 4y$.
6. $8y = x^3$ and $y = 2x$.

7. $x^2 - y^2 = 5$ and $2x - y = 1$.
8. $y^2 = x^3$ and $7x - 3y = 4$.
9. $xy = 8$ and $x^2 + y^2 = 20$.
10. $x^2 + y^2 = 9$ and $y^2 = 2x - 6$.
11. $x^2 + y^2 = 9$ and $y^2 = x + 3$.
12. $x^2 + y^2 = 4$ and $y^2 = 2x + 5$.
13. $4x^2 + y^2 = 4$ and $4x^2 - 4y^2 + 11 = 0$.
14. $y^3 = 16x^2$ and $3y = x^2 + 8$.
15. $x + y = xy$ and $x^2 + y^2 = 8$.

Find the points of intersection of the graphs of the following polar equations and check the results graphically :

16. $\rho^2 = 4$ and $\rho = 2 \tan \theta$.
17. $\rho = 4(1 + \cos \theta)$ and $\rho \cos \theta = 3$.
18. $\rho = 2(1 - \cos \theta)$ and $\rho = 2/(1 - \cos \theta)$.
19. $\rho = a \sin \theta$ and $\rho \sin \theta = a$.
20. $\rho = 4 \sin^2 \theta$ and $\rho = 3$.

THE SECOND FUNDAMENTAL PROBLEM

45. To find the equation of a given locus. The method of finding the equation corresponding to a given locus will vary from problem to problem. In general, the following steps should be taken in order :

(a) *Choose a suitable set of axes.* If the problem does not fix the axes, any suitable lines may be chosen. Generally some line of the figure suggested by the problem taken as an axis or some fixed point as origin will serve best. The best selection of axes is that which gives the simplest equation.

(b) *Locate all data with respect to these axes.*

(c) *Locate approximately on the graph a point satisfying the conditions of the problem and call its coördinates (x, y) .*

(d) *Draw any auxiliary lines suggested by the conditions of the problem.*

(e) *Out of the figure thus constructed obtain an equation involving the coördinates (x, y) of the point whose locus is given.*

Relations of the sides of a right triangle, similar triangles, related slopes, and related distances are some of the possibilities for use in forming the desired equation.

Having found the equation, we may use it to aid in plotting the locus.

Similar steps are to be followed in finding the polar equation of a locus.

A locus is defined by giving some characteristic geometric property of it. From this property we derive an equation connecting the coördinates of any point on the locus. Then every other property of the locus is involved in the equation and may be obtained by an analysis of the equation.

EXAMPLE 1. Find the equation of the locus of points which are equidistant from the line $4x + 3y + 25 = 0$ and from the point $(4, 3)$.

Draw the line and locate the point on a set of axes. Select $P(x, y)$ any point on the locus and draw PA and PB . From the problem

$$PA = BP.$$

$$\text{But } PA = \sqrt{(x-4)^2 + (y-3)^2} \text{ and } BP = \frac{4x + 3y + 25}{-5}.$$

Hence the required equation is

$$\frac{4x + 3y + 25}{-5} = \sqrt{(x-4)^2 + (y-3)^2}.$$

Squaring, clearing of fractions, and collecting terms, we have

$$9x^2 - 24xy + 16y^2 - 400x - 300y = 0.$$

EXAMPLE 2. Find the equation of the locus of a point which moves so that it is always three times as far from one end of a line segment as from the other, where the line segment is eight units long.

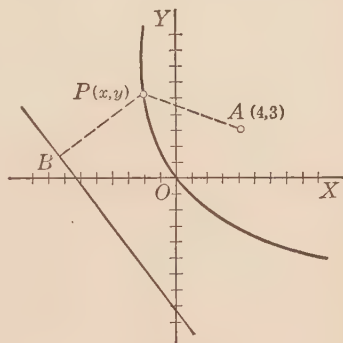


FIG. 40

If we choose the midpoint of the line segment as origin and the line including the segment as the x -axis, the coördinates

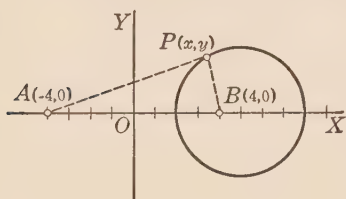


FIG. 41

of the end points are $A(-4, 0)$ and $B(4, 0)$. By the condition of the problem $AP = 3 BP$ or $BP = 3 AP$. We shall solve the problem for $AP = 3 BP$. Let (x, y) be the coördinates of P .

$$\text{Then } AP = \sqrt{(x+4)^2 + y^2}$$

$$\text{and } BP = \sqrt{(x-4)^2 + y^2}.$$

Hence the equation of the locus of P is

$$\sqrt{(x+4)^2 + y^2} = 3\sqrt{(x-4)^2 + y^2}.$$

Squaring and reducing, we may write the equation in the form

$$x^2 + y^2 - 10x + 16 = 0.$$

The locus may be graphed from this equation.

A number of the succeeding chapters further illustrate the problem of finding the equation of a locus.

EXERCISES

1. Find the equation of the locus of points at a distance of 10 units from the origin. Draw the locus.
2. Find the equation of the locus of points at a distance of 5 units from the point $(3, 4)$. Draw the locus.
3. Find the equation of the locus of a point the product of whose distances from two fixed perpendicular lines equals a constant.
4. Find the equation of the locus of a point whose distance from the origin is numerically equal to the slope of the line joining it to the origin.
5. A point moves so that the square of its distance from $(0, 3)$ diminished by the square of its distance from $(4, 0)$ is equal to the square of the distance between the two given points. Find the equation of the locus and show its position relative to the given points.

6. Find the equation of the locus of the points equidistant from the y -axis and the point $(4, 0)$. Draw the locus.

7. Find the equation of the locus of the points equidistant from the line $y = -2$ and the point $(2, 0)$. Draw the locus.

8. A point moves so that the sum of its distances from $(3, 0)$ and $(-3, 0)$ is equal to 10. Find the equation of its locus and draw the graph.

9. A point moves so that the difference of its distances from $(5, 0)$ and $(-5, 0)$ is equal to 8. Find the equation of its locus and draw the graph.

10. A point moves so that its distance from the line $x = -2$ is twice its distance from the point $(1, 0)$. Derive the equation of its locus and draw the graph.

11. A point moves so that its distance from the point $(0, 2)$ is twice its distance from the line $y = -1$. Derive the equation of its locus and draw the graph.

12. Find the equation of the locus of the points the sum of whose distances from $(1, 1)$ and $(-1, -1)$ is 4.

13. Find the equation of the locus of the points equidistant from $(-\sqrt{2}, \sqrt{2})$ and the line $x - y = 2\sqrt{2}$.

14. A point moves so that the slope of the line joining it to $(-1, -3)$ equals three times the slope of the line joining it to $(1, 1)$. Find the equation of its locus.

15. A point moves so that the product of its distances from two adjacent sides of a square plus the product of its distances from the other two sides of the square is equal to the area of the square. Find the equation of the locus and show its position relative to the square.

46. Change of axes. It sometimes happens that the choice of axes made at the beginning of the solution of a problem does not give the simplest form of the equation. Two types of change of axes are in common use. They are called **translation of axes** and **rotation of axes**; they will be described and illustrated in the next two sections.

47. Translation. Translation of axes is a change from one set of axes to a new set parallel to the old axes. In Fig. 42 we shall express the coördinates (x, y) of the point P in terms of the coördinates (x', y') of the same point referred to

the new axes $X'O'Y'$. The coördinates of O' are (h, k) . From the figure we have

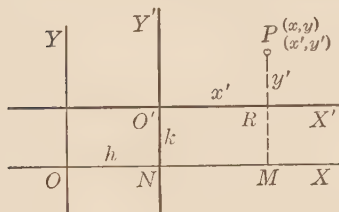


FIG. 42

$$x = OM = ON + NM = h + x'$$

and

$$y = MP = MR + RP = k + y',$$

$$\text{or } x = x' + h \text{ and } y = y' + k.$$

The necessary substitutions for changing from the new axes

back to the old are obtained by solving for x' and y' ; they are

$$x' = x - h \text{ and } y' = y - k.$$

EXAMPLE 1. In Example 2, Sec. 45, transform the equation $x^2 + y^2 - 10x + 16 = 0$ by translating the axes so that the origin is at $(5, 0)$.

The equations of translation are

$$x = x' + 5 \text{ and } y = y'.$$

Substituting these values, we obtain

$$(x' + 5)^2 + y'^2 - 10(x' + 5) + 16 = 0,$$

or

$$x'^2 + y'^2 = 9,$$

thus changing the form of the equation but not changing the locus. By reference to the expression for the distance between two points, it is seen that the last equation states that the square of the distance from (x', y') to the origin is 9. Hence the point (x', y') lies on a circle of radius 3 with the center at the origin of the new axes, or with center at the point $(5, 0)$ referred to the original axes.

EXAMPLE 2. Translate the axes to a new set of axes in such a way as to remove the x and y terms from the equation $x^2 + y^2 - 4x + 6y - 12 = 0$.

First method. Let $x = x' + h$ and $y = y' + k$ and collect terms. The result is

$$x'^2 + y'^2 + (2h - 4)x' + (2k + 6)y' + h^2 + k^2 - 4h + 6k - 12 = 0.$$

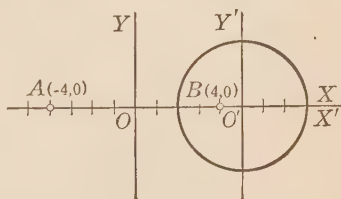


FIG. 43

We may now choose h and k so that the coefficients of x' and y' are each zero. Letting

$$2h - 4 = 0 \quad \text{and} \quad 2k + 6 = 0,$$

we have $h = 2$ and $k = -3$. Substituting these values, we obtain

$$x'^2 + y'^2 - 25 = 0$$

as the equation of the locus referred to new axes, chosen so that there are no x' or y' terms in the equation. The new origin is at $(2, -3)$.

Second method. Completing squares in the equation $x^2 + y^2 - 4x + 6y - 12 = 0$, we may write

$$(x - 2)^2 + (y + 3)^2 - 25 = 0.$$

If we replace $x - 2$ by x' and $y + 3$ by y' , we have

$$x'^2 + y'^2 - 25 = 0.$$

Hence the two methods of determining the translation lead to the same equation.

48. Rotation. Rotation of axes is changing from one set of axes to a new set which can be obtained by rotating the old axes about the origin. The problem is to express the coördinates (x, y) of a point P in terms of the coördinates (x', y') of the same point with respect to the axes $X'OY'$, which make the angle θ with the axes XOY . Drawing such auxiliary lines as are indicated in the figure, we see that

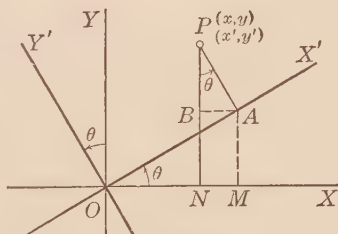


FIG. 44

$$\begin{aligned} x &= ON = OM - NM = OM - BA \\ &= OA \cos \theta - AP \sin \theta = x' \cos \theta - y' \sin \theta, \end{aligned}$$

or $x = x' \cos \theta - y' \sin \theta,$

and $y = NP = NB + BP = MA + BP$
 $= OA \sin \theta + AP \cos \theta$
 $= x' \sin \theta + y' \cos \theta,$

or $y = x' \sin \theta + y' \cos \theta.$

EXAMPLE. In Example 1, Sec. 45, transform the equation of the locus $9x^2 - 24xy + 16y^2 - 400x - 300y = 0$ by rotating the axes through an angle θ such that $\sin \theta = \frac{3}{5}$ and $\cos \theta = \frac{4}{5}$.

The equations of rotation are

$$x = \frac{4}{5}x' - \frac{3}{5}y' \quad \text{and} \quad y = \frac{3}{5}x' + \frac{4}{5}y'.$$

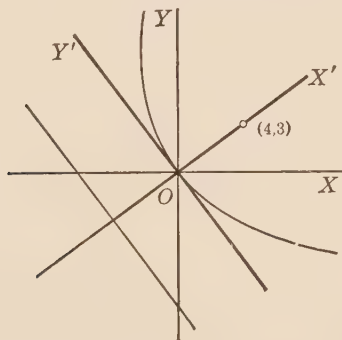


FIG. 45

Making these substitutions in the equation, we have

$$\begin{aligned} 9\left(\frac{4x' - 3y'}{5}\right)^2 - 24\left(\frac{4x' - 3y'}{5}\right)\left(\frac{3x' + 4y'}{5}\right) + 16\left(\frac{3x' + 4y'}{5}\right)^2 \\ - 400\left(\frac{4x' - 3y'}{5}\right) - 300\left(\frac{3x' + 4y'}{5}\right) = 0. \end{aligned}$$

Reducing, we have

$$y'^2 - 20x' = 0.$$

This is the equation of the same curve referred to the new axes.

EXERCISES

1. Simplify $x^2 + y^2 + 4x - 6y - 12 = 0$ by translating the origin to $(-2, 3)$.

2. Simplify $y^2 - 8y - 2x + 18 = 0$ by translating the origin to $(1, 4)$.

3. What do the equations $2x + y - 1 = 0$ and $5x + 3y - 5 = 0$ become if the origin is translated to the point of intersection of their loci?

4. To what point must the origin be translated so that the equation of the locus $x^2 + y^2 + 10x - 3y - 5 = 0$ shall have no first degree terms?

5. The origin is translated to any point (h, k) on the line $Ax + By + C = 0$. What is the new equation?

6. Graph $y = (x - 2)^3$ after translating the origin to $(2, 0)$.

7. Graph $(y - 1)^2 = (x + 3)^3$ after translating the origin to $(-3, 1)$.

8. Graph $(y - 1)(x - 2) + 7 = 0$ after translating the origin to $(2, 1)$.

9. Transform $x^2 - y^2 = a^2$ by rotating the axes through -45° .

10. Transform $2xy = a^2$ by rotating the axes through 45° .

11. Simplify the equation derived in Exercise 12, page 61, by rotating the axes through 45° .

12. Simplify the equation derived in Exercise 13, page 61, by rotating the axes through -45° .

13. Simplify the equation $41x^2 - 84xy + 104y^2 = 500$ by rotating the axes through the acute angle $\tan^{-1}\frac{1}{2}$.

49. Graph of a function. The graph may be used to illustrate the relation between the value of a function and the value of the variable upon which the function depends. The graph may be used for this purpose even if it is impossible to write this relation by means of an equation. If we use ten-year periods beginning with 1790 as abscissas and the census returns as ordinates of points, a curve through these points will illustrate the population trend in this country. We shall be interested in this book, however, in those functions which have an analytic representation, or for which an approximate analytic representation can be found, that is, in those functions in which the relation between the function and the variable can be expressed in the form of an equation.

EXAMPLE 1. Graph the function $2x^2 - 3x + 1$.

In algebra we were chiefly interested in the zeros of such an expression. Let $y = 2x^2 - 3x + 1$ and graph. The curve

(Fig. 46) not only shows the zeros of the function, but also shows for what values of x the function is positive and for what values of x it is negative and gives some idea of the change in the function due to the change in x .

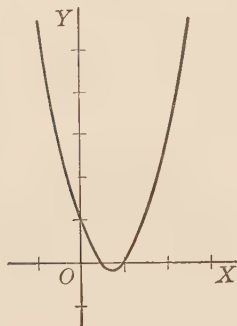


FIG. 46

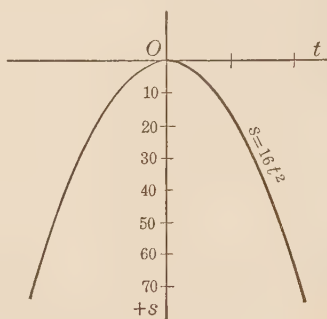


FIG. 47

EXAMPLE 2. Show graphically the relation between the time t and the distance s a body would fall in a vacuum.

From physics we have $s = \frac{1}{2}gt^2$, s representing distance in feet and t representing time in seconds. Choosing the horizontal axis for the time axis and the vertical axis for the distance axis with positive s measured downward, we construct the graph as shown in Fig. 47, using $g = 32$. The part of the curve to the right of the s -axis represents the relation in which we are interested. The part of the curve to the left

of the s -axis represents the relation that is shown by the equation but does not represent the physical problem.

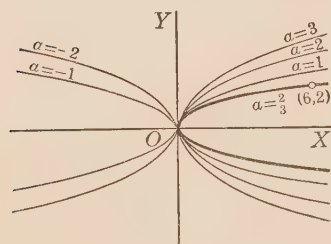


FIG. 48

50. Families of curves. EXAMPLE. Graph $y^2 = ax$.

If we assign a definite value to a , say $a = 1$, the equation $y^2 = x$ can be graphed as in Fig. 48. By assigning other values to a , $a = 2, 3, -1, -2$, etc., the corresponding curves can be graphed. The curves are all of the same general shape.

Each depends upon the value given to the arbitrary constant, or parameter, a . Such a system of curves, derived from a single equation by giving different values to the parameter, is called a **family of curves**. If we want the curve of this family which passes through some given point, for example $(6, 2)$, it is necessary to determine a so that the coördinates $(6, 2)$ satisfy the equation. Substituting these coördinates in the equation $y^2 = ax$, we have

$$4 = 6a.$$

Hence a must equal $\frac{2}{3}$, and the desired equation is

$$y^2 = \frac{2}{3}x.$$

The graph is shown in Fig. 48 as the curve passing through $(6, 2)$.

MISCELLANEOUS EXERCISES

1. Discuss and draw the graphs of the following equations:

(a) $2x = 3 + 5y$.

(h) $y - 1 = x^3$.

(b) $x^2 - 4y^2 = 0$.

(i) $y^2 = (x - 1)^3$.

(c) $x^2 - 4x + 4 = 0$.

(j) $y = \frac{3}{(x - 3)}$.

(d) $x^2 + y^2 = 13$.

(e) $3x^2 + y^2 = 12$.

(k) $x = \frac{2y}{(y + 1)}$.

(f) $x^2 - 4y^2 + 4 = 0$.

(g) $x^2 = 6y - 6$.

(l) $y^2 = (x - 2)(x - 1)^2$.

2. Find the points of intersection of (a) $y = 4x/(x^2 + 3)$ and $3y = x$; (b) $y = 4x/(x^2 + 3)$ and $x - 2y + 1 = 0$. Draw the figures.

3. Find the equation of the line through the points of intersection of $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x - 4y = 0$.

4. Show that $y^2 - y - 6 = 0$ and $x^2 - 2xy + y^2 - x + y - 12 = 0$ represent lines which form a parallelogram, and find (a) the lengths of its sides; (b) the angle of intersection of its diagonals.

5. Show that $Ax^2 + Bxy + Cy^2 = 0$ represents (a) a pair of distinct lines intersecting at the origin if $B^2 - 4AC > 0$; (b) two coincident lines if $B^2 - 4AC = 0$; (c) an imaginary locus if $B^2 - 4AC < 0$.

6. What is the condition under which $Ax^2 + Bxy + Cy^2 = 0$ represents two perpendicular lines?

7. Show that the following equations represent either isolated points or no real loci:

- | | |
|------------------------------------|-------------------------------------|
| (a) $x^2 + y^2 = 0$. | (d) $(x + 3)^2 + (y - 4)^2 = 0$. |
| (b) $x^2 + 4 = 0$. | (e) $x^2 + 2xy + y^2 + 1 = 0$. |
| (c) $(x - 1)^2 + (y + 2)^2 = -1$. | (f) $(x^2 - 1)^2 + (y - 1)^2 = 0$. |

8. Reduce $x^4 - y^2 + x^2y^2 = 0$ to polar form and draw the curve.

9. Reduce $\rho = a(\cos \theta + \sin \theta)$ to rectangular form.

10. Show that $a^2y^2 = (x^2 + y^2)(x^2 + y^2 - 2ax)$ can be reduced to the form $\rho = a(\cos \theta \pm 1)$.

11. Graph $f(x) = 2x^2 - 5x + 3$ and show from the graph for what values of x , $f(x)$ is positive; zero; negative.

12. Illustrate graphically that $f(x) = x^2 + x + 1$ is always greater than zero.

13. Illustrate graphically for what values of x the function $f(x) = (x + 2)(x - 1)(x - 5)$ is greater than zero; less than zero.

14. Two masses m_1 and m_2 at a distance of r units apart attract each other with a force F given by the equation $F = m_1m_2/r^2$. If $m_1 = 2$ and $m_2 = 5$, draw a graph showing the relation between F and r .

15. If a stone is thrown upward from the earth's surface with an initial velocity of 80 feet per second, the distance s (in feet) traversed in t seconds is given by the equation $s = 80t - \frac{1}{2}gt^2$. Assuming $g = 32$, illustrate graphically the relation between s and t .

16. According to Boyle's law the pressure and volume of a gas at a constant temperature are connected by the equation $PV = k$. Graph this equation for $k = 20$.

17. Draw several members of each of the following families of curves:

- | | | |
|---------------------|-------------------------|--------------------|
| (a) $y = x^2 + c$. | (c) $x^2 + y^2 = c^2$. | (e) $x^cy = 2$. |
| (b) $xy = c^2$. | (d) $y = x^c$. | (f) $y^2 = cx^3$. |

18. Determine in each of the cases in Exercise 17 the value of c for which a member of the family will pass through the point $(3, 4)$.

19. Show that $f_1(x, y) + k \cdot f_2(x, y) = 0$ represents a family of curves passing through all the points of intersection of the two curves $f_1(x, y) = 0$ and $f_2(x, y) = 0$.

20. Write the equation of the family of curves passing through the points of intersection of $y^2 = 8x$ and $y^2 = 12x - 8$. Find the equation of that member of the family which is a straight line.

21. Find the equation of the locus of a point if lines drawn from it to two fixed points include an angle of 45° .

22. The slope of a line passing through $(-a, 0)$ is n times the slope of a line passing through $(a, 0)$. Determine the nature of the locus of their point of intersection. Can n be chosen so that the locus will pass through $(a, 0)$? $(0, 0)$? $(-a, 0)$?

23. Show that the equation $x^2 + y^2 = a^2$ is unchanged by a rotation of the axes.

CHAPTER IV

THE CIRCLE

51. Equation of a circle. A circle is the locus of points in a plane which are equidistant from a given point called the center.

Let (h, k) be the center of a circle of radius r . From the definition the distance from any point $P(x, y)$ on the circle to the center (h, k) is r . Hence the equation is

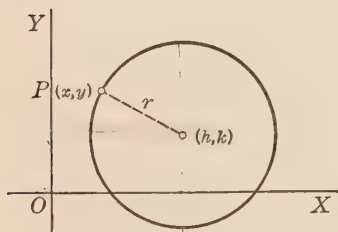


FIG. 49

$$\sqrt{(x - h)^2 + (y - k)^2} = r,$$

$$\text{or } (x - h)^2 + (y - k)^2 = r^2. \quad (1)$$

This may be written in the form

$$x^2 + y^2 + ax + by + c = 0, \quad (2)$$

where $a = -2h$, $b = -2k$ and $c = h^2 + k^2 - r^2$.

Any circle is determined by its center and radius. Hence any circle can be represented by an equation like (2). For example, the circle whose radius is 5 and center $(-2, 3)$ is represented by the equation $x^2 + y^2 + 4x - 6y - 12 = 0$.

If the center of the circle is at the origin, $h = k = 0$, and equation (1) becomes

$$x^2 + y^2 = r^2.$$

52. Theorem. *The equation*

$$Ax^2 + Ay^2 + Bx + Cy + D = 0, \quad (A \neq 0) \quad (3)$$

in which the coefficients of x^2 and y^2 are equal and in which there is no xy term, represents a circle.

Dividing equation (3) by A , we have an equation of the form

$$x^2 + y^2 + ax + by + c = 0,$$

where $a = B/A$, $b = C/A$, and $c = D/A$. Completing the squares in the left member and transposing c , we have

$$x^2 + ax + \frac{a^2}{4} + y^2 + by + \frac{b^2}{4} = \frac{a^2}{4} + \frac{b^2}{4} - c,$$

or
$$\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} - c. \quad (4)$$

Equation (4) is the condition that the distance from the point (x, y) to the fixed point $(-a/2, -b/2)$ is a constant. This condition is satisfied by any point (x, y) which lies on the circle whose center is $(-a/2, -b/2)$ and whose radius is $\sqrt{\frac{a^2}{4} + \frac{b^2}{4} - c}$, and by no other point. Hence equation (3) represents a circle.

If $a^2/4 + b^2/4 - c = 0$, equation (4) is satisfied by the coordinates of only one real point, $(-a/2, -b/2)$, and the locus is sometimes called a **point circle**.

If $a^2/4 + b^2/4 - c < 0$, there is no real locus.

EXAMPLE. Find the center and radius of the circle

$$x^2 + y^2 - 8x + 10y - 4 = 0.$$

Completing squares and rearranging, we may write

$$(x - 4)^2 + (y + 5)^2 = 45.$$

Comparing with equation (1), we see that the center of the circle is $(4, -5)$ and the radius is $r = \sqrt{45} = 3\sqrt{5}$. The circle should be drawn with the center and radius just found.

EXERCISES

1. Find the equations of the following circles and draw a figure in each case :

- (a) With radius 3 and with center at the origin.
- (b) Passing through $(2, 4)$ and with center at the origin.
- (c) Passing through the origin and with center at $(2, 4)$.
- (d) Passing through $(0, 2)$ and with center at $(-2, 5)$.
- (e) With ends of a diameter at $(-3, 2)$ and $(4, -5)$.
- (f) Touching the x -axis and with center at $(3, 4)$.
- (g) Touching the line $2x = 3$ and with center at $(-1, \frac{1}{2})$.

2. Write the equation of the circle of radius r tangent to both axes and lying in the (a) first quadrant; (b) second quadrant; (c) third quadrant; (d) fourth quadrant.

3. Find the equation of the circle which has for diameter that segment of the line $12x - 5y = 60$ included between the coordinate axes.

4. Reduce the following equations to the form of equation (1) (Sec. 51), determine the center and the radius, and draw the circle:

(a) $x^2 + y^2 - 2x - 6y - 6 = 0$. (d) $x^2 + y^2 = 12x$.

(b) $x^2 + y^2 + 10x - 4y + 4 = 0$. (e) $3x^2 + 3y^2 + 4y - 7 = 0$.

(c) $2x^2 + 2y^2 - 3x + 5y = 3$. (f) $x^2 + y^2 = x + y$.

5. Write the equation of the line of centers (the straight line containing the centers) of the circles $x^2 + y^2 - 6x + 5 = 0$ and $2x^2 + 2y^2 - 2x + 6y + 3 = 0$.

6. What form does equation (2) (Sec. 51) assume if the circle (a) passes through the origin? (b) has its center on the x -axis? (c) has its center on the y -axis? (d) has its center on the line $y = x$?

7. Write the equations of the following families of circles:

(a) With center at the origin.

(b) Passing through the origin and with center on the x -axis.

(c) Passing through the origin and with center on the line $y = x$.

(d) Touching the lines $y = \pm 6$.

8. If two circles are concentric, how do their equations differ?

9. Write the equation of the circle passing through (4, 2) and concentric with the circle $x^2 + y^2 + 5x - 6y + 6 = 0$.

10. Show that the locus of points twice as far from (0, 3) as from (6, 0) is a circle and show its position relative to the given points.

11. Show that the locus of points the sum of the squares of whose distances from each of the four vertices of a square is equal to four times the area of the square is a circle circumscribing the square.

12. Prove analytically that an angle inscribed in a semicircle is a right angle.

13. Prove analytically that the radius perpendicular to a chord of a circle bisects the chord.

14. Prove analytically that all angles inscribed in the same segment of a circle are equal.

15. Show that the following equations represent either point circles or no real loci :

$$(a) x^2 + y^2 = 0. \qquad (c) x^2 + y^2 - 2x - 2y + 2 = 0.$$

$$(b) x^2 + y^2 + 4 = 0. \qquad (d) x^2 + y^2 - 2x - 2y + 3 = 0.$$

53. Circle determined by three conditions. It will be observed that the general equation of the circle contains three independent constants. A circle can, therefore, be found that will satisfy three suitable conditions; for example, that will pass through three points not in a straight line. Suppose that (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) are three points not in the same straight line. The equation of any circle may be written in the form

$$x^2 + y^2 + ax + by + c = 0.$$

If this circle passes through the three given points, the equation must be satisfied by the coördinates of each point. Substituting in turn the coördinates of the three points, we get the equations

$$x_1^2 + y_1^2 + ax_1 + by_1 + c = 0,$$

$$x_2^2 + y_2^2 + ax_2 + by_2 + c = 0,$$

$$x_3^2 + y_3^2 + ax_3 + by_3 + c = 0;$$

these will determine a , b , and c when solved simultaneously.

The equation of the circle may be used in the form

$$(x - h)^2 + (y - k)^2 = r^2,$$

in which case we have

$$(x_1 - h)^2 + (y_1 - k)^2 = r^2,$$

$$(x_2 - h)^2 + (y_2 - k)^2 = r^2,$$

$$(x_3 - h)^2 + (y_3 - k)^2 = r^2.$$

The simultaneous solution of these equations determines h , k , and r .

EXAMPLE 1. Find the equation of the circle through $(-2, -2)$, $(10, -8)$, and $(7, 1)$.

The three equations expressing the condition that the circle $x^2 + y^2 + ax + by + c = 0$ goes through the three points $(-2, -2)$, $(10, -8)$, and $(7, 1)$ are

$$4 + 4 - 2a - 2b + c = 0,$$

$$100 + 64 + 10a - 8b + c = 0,$$

$$49 + 1 + 7a + b + c = 0.$$

Solving simultaneously, we find $a = -8$, $b = 10$, and $c = -4$. This gives the required equation,

$$x^2 + y^2 - 8x + 10y - 4 = 0.$$

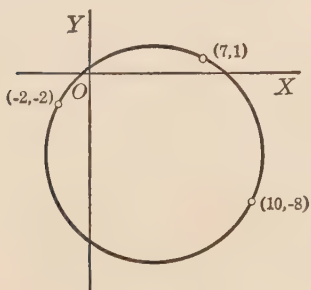


FIG. 50

EXAMPLE 2. Find the equation of the circle which is tangent to the line $3x - 4y - 20 = 0$, which has its center on the line $y = 2x$, and which passes through the point $(-2, -2)$.

The equation which expresses the condition that the center (h, k) of the circle is on the line $y = 2x$ is

$$k = 2h.$$

Since the circle is tangent to the line $3x - 4y - 20 = 0$, the distance from the line to the center is r . This condition is

$$\frac{3h - 4k - 20}{5} = -r,$$

in which $-r$ is used since the center is on the same side of the line as the origin. The condition that the circle passes through $(-2, -2)$ may be written

$$(-2 - h)^2 + (-2 - k)^2 = r^2.$$

Solving the three equations simultaneously, we find two sets of values for h , k , and r ; namely, $h = 1$, $k = 2$, $r = 5$ and

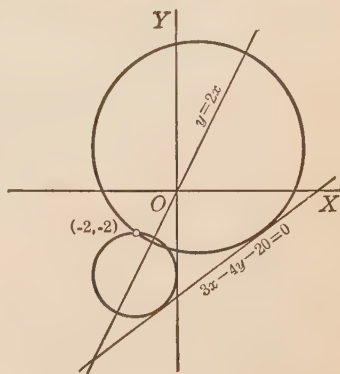


FIG. 51

$h = -2$, $k = -4$, $r = 2$. Hence there are two circles satisfying the conditions of the problem. The equations of the two circles are

$$(x-1)^2 + (y-2)^2 = 25$$

and

$$(x+2)^2 + (y+4)^2 = 4.$$

EXERCISES

1. Write the equation of the circle passing through

(a) $(1, 1)$, $(3, 2)$, and $(2, -1)$.

(b) $(-2, 0)$, $(0, 3)$, and $(1, -2)$.

(c) $(0, 0)$, $(-2, -1)$, and $(4, 5)$.

(d) $(-\frac{1}{2}, 2)$, $(3, \frac{1}{4})$, and $(0, -2)$.

2. Write the equation of the circle circumscribing the triangle whose vertices are $(1, 3)$, $(2, 1)$, and $(7, -1)$.

3. Find the equation of the circle circumscribing the triangle whose sides are formed by the lines $3x + y = 25$, $x - 2y + 1 = 0$, and $x - y + 5 = 0$.

4. Find the equation of the circle with center on the y -axis and passing through $(0, -2)$ and $(3, 7)$.

5. The x -axis cuts a circle in the points $(1, 0)$ and $(5, 0)$. Find the equation of the circle if the center is on the line $y = 2x$.

6. Write the equation of the circle passing through $(7, 3)$ and $(-3, -7)$ if the center is on the line $x + 2y = 6$.

7. Write the equation of the circle with center at $(8, 7)$ and touching the line $3x + 4y - 12 = 0$.

8. Write the equation of the circle passing through $(-1, 6)$ and touching the y -axis at $(0, 3)$.

9. Write the equations of the two circles touching the line $x - y + 2 = 0$ with centers on the x -axis and with radii $4\sqrt{2}$.

10. Write the equation of the circle passing through $(1, 6)$ and touching the line $x - 2y + 1 = 0$ at $(-1, 0)$.

11. Write the equations of the two circles with radii 10 and passing through $(-2, 3)$ and $(6, -1)$.

12. Find the equations of the two circles touching both axes and passing through $(4, 2)$.

13. Find the equations of the two circles touching the x -axis and passing through $(6, 2)$ and $(10, 10)$.

14. Write the equations of the two circles passing through (1, 2) and (3, 4) and touching the line $3x + y - 3 = 0$.

15. Find the equation of the circle inscribed in the triangle formed by the lines $x + y - 15 = 0$, $x - 7y - 11 = 0$, and $17x + 7y + 65 = 0$.

54. Circles through the intersection of two circles. The equation

$$(x^2 + y^2 + a_1x + b_1y + c_1) + k(x^2 + y^2 + a_2x + b_2y + c_2) = 0 \quad (5)$$

represents a circle through the points of intersection of the circles

$$x^2 + y^2 + a_1x + b_1y + c_1 = 0 \quad \text{and} \quad x^2 + y^2 + a_2x + b_2y + c_2 = 0$$

for every value of k except for $k = -1$. Equation (5) represents a circle because it can be put in the form of equation (3)

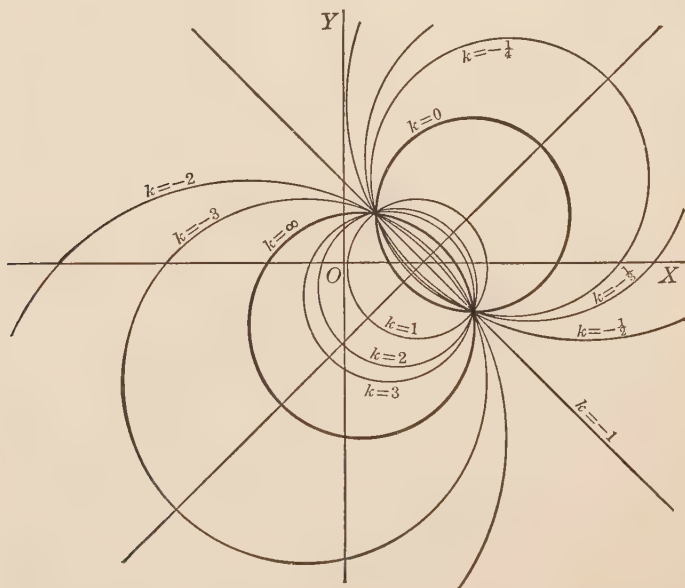


FIG. 52

(Sec. 52). It will pass through the points of intersection of the two circles for every k , since the coördinates of the points of intersection make each expression in parenthesis

equal to zero. Equation (5) contains the parameter k . By the proper choice of k we can make the circle (5) satisfy one more suitable condition. It should be kept in mind that equation (5) already satisfies two conditions, since the circle represented by (5) passes through the two points of intersection of two circles.

In case $k = -1$, equation (5) reduces to

$$(a_1 - a_2)x + (b_1 - b_2)y + c_1 - c_2 = 0.$$

This is the equation of the line through the points of intersection of the two circles. The common chord is a segment of this line. If the circles do not intersect, the line does not intersect either circle. In either case the line is called the **radical axis**.

Members of the family of curves

$$x^2 + y^2 - 16x - 6y + 37 + k(x^2 + y^2 - 2x + 8y - 33) = 0$$

through the points of intersection of the circles

$$x^2 + y^2 - 16x - 6y + 37 = 0$$

and

$$x^2 + y^2 - 2x + 8y - 33 = 0$$

are drawn in the accompanying figure for $k = -3, -2, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, 1, 2$, and 3 .

55. Distance from a point to a circle along a tangent. From a point $P(x', y')$ outside the circle (Fig. 53)

$$(x - h)^2 + (y - k)^2 - r^2 = 0 \quad (6)$$

draw the tangent PT and the line PC . The triangle CTP has a right angle at T . Hence

$$\overline{PT}^2 = \overline{PC}^2 - \overline{CT}^2 = (x' - h)^2 + (y' - k)^2 - r^2.$$

We observe that the expression for $\overline{PT}^2$ can be obtained by substituting x' and y' for x and y , respectively, in the left member of equation (6). Therefore, if the coördinates of a point outside the circle are substituted in the left member of the equation of the circle in the form (6), that expression is no longer zero, but is the square of the distance from

the point to the circle measured along the tangent. The equation

$$x^2 + y^2 + ax + by + c = 0,$$

being a rearrangement of

$$(x - h)^2 + (y - k)^2 - r^2 = 0,$$

may be used instead of the latter form to find this distance.

Assume that (x', y') is a point from which the distances to the two circles

$$x^2 + y^2 + a_1x + b_1y + c_1 = 0,$$

$$x^2 + y^2 + a_2x + b_2y + c_2 = 0$$

measured along the tangents are equal. Then

$$\begin{aligned} x'^2 + y'^2 + a_1x' + b_1y' + c_1 \\ = x'^2 + y'^2 + a_2x' + b_2y' + c_2, \end{aligned}$$

$$\begin{aligned} \text{or } (a_1 - a_2)x' + (b_1 - b_2)y' \\ + c_1 - c_2 = 0. \end{aligned}$$

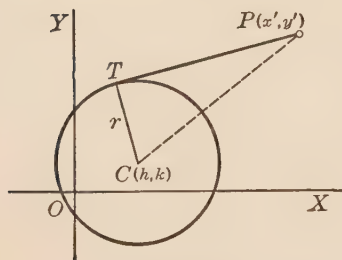


FIG. 53

But this is the condition that the point (x', y') lies on the radical axis. Hence *the radical axis of two circles is the locus of all points from which tangents of equal length may be drawn to the two circles.*

EXERCISES

1. Write the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 + 4x - 4y - 17 = 0$ and $x^2 + y^2 - 8x + 2y + 7 = 0$ and passing through (a) the point $(4, 3)$; (b) the point $(0, 0)$.

2. Write the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - 8x - 4y - 16 = 0$ and $x^2 + y^2 + 2x + 6y - 16 = 0$ and passing through the center of the first circle.

3. Write the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - x - 5y + 2 = 0$ and $x^2 + y^2 - 7x - y - 3 = 0$ and having its center on (a) the y -axis; (b) the line $x - y = 0$.

4. Find the equations of the common chords of the given pairs of circles in Exercises 1, 2, and 3.

5. Find the length of the common chord of the two circles $x^2 + y^2 - 2x - 8y - 8 = 0$ and $x^2 + y^2 + 2x + 20y - 24 = 0$.

6. Find the length of the tangent line from $(3, 7)$ to each of the circles $x^2 + y^2 - 3x + 5y - 4 = 0$ and $x^2 + y^2 - 19x - 3y + 100 = 0$. Draw a figure.

7. Find the equation of the radical axis of the two circles in Exercise 6.

8. Find the length of the tangent line from $(4, 1)$ to each of the circles $x^2 + y^2 - 6x - 6y + 9 = 0$ and $x^2 + y^2 - 7x - 5y + 12 = 0$. Discuss the meaning of the result.

9. Find the equation of the radical axis of the circles in Exercise 8. Draw a figure.

10. Find the length of the tangent line from $(2, 3)$ to each of the circles $x^2 + y^2 + 2x - 18y + 37 = 0$ and $x^2 + y^2 - 6x - 2y + 5 = 0$. Discuss the meaning of the result.

11. Find the equation of the radical axis of the circles in Exercise 10. Draw a figure.

12. Show that the circles $x^2 + y^2 - 2x - 4y - 11 = 0$ and $x^2 + y^2 - 12x + 6y + 19 = 0$ intersect in two distinct points.

13. Show that the circle $x^2 + y^2 = 20$ touches the circle $x^2 + y^2 - 12x + 6y = 80$.

14. Show that the circles $2x^2 + 2y^2 + 2x - 12y + 3 = 0$ and $x^2 + y^2 - 6x - 2y + 9 = 0$ do not intersect.

15. Show that the radical axis of $x^2 + y^2 + a_1x + b_1y + c_1 = 0$ and $x^2 + y^2 + a_2x + b_2y + c_2 = 0$ is perpendicular to the line of centers of the two circles.

16. Show that the radical axes of the following circles taken in pairs meet in a common point: $x^2 + y^2 + 6x - 8y = 0$, $x^2 + y^2 - 14x + 40 = 0$, and $x^2 + y^2 - 6x + 16y + 72 = 0$.

56. Polar equation of a circle. Let (ρ_1, θ_1) be the polar coördinates of the center and (ρ, θ) the polar coördinates of any point on a circle of radius r . Equating the square of the distance from (ρ_1, θ_1) to (ρ, θ) to r^2 , we have for the polar equation of a circle

$$\rho^2 + \rho_1^2 - 2\rho\rho_1 \cos(\theta - \theta_1) = r^2.$$

The polar equation is, in general, more complicated than the equation of the circle in rectangular coördinates. Some

from Law of Cosines.

special cases of the equation of the circle in polar coordinates are quite simple and useful.

(a) If the center of the circle is at the pole, then $\rho_1 = 0$ and the equation reduces to

$$\rho = r.$$

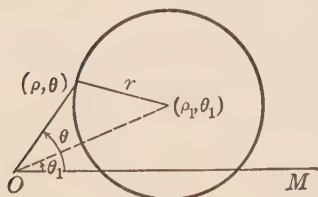


FIG. 54

(b) If the pole is at the end of a diameter that extends to the right along the polar axis, then $\theta_1 = 0$ and $\rho_1 = r$ and the equation becomes

$$\rho = 2 r \cos \theta.$$

(c) If the pole is at the end of a diameter that is perpendicular to the polar axis and extends upward, then $\theta_1 = 90^\circ$ and $\rho_1 = r$ and the equation becomes

$$\rho = 2 r \sin \theta.$$

EXERCISES

1. Show directly from a figure that $\rho = a \cos \theta$ represents a circle with diameter a .

2. By changing the equation to rectangular coordinates show that $\rho = a \cos \theta$ represents a circle.

3. Prove that $\rho = a \sin \theta$ represents a circle and show its position relative to the pole and the polar axis.

4. Draw the following circles :

(a) $\rho = 10 \cos \theta.$ (c) $\rho = -8 \cos \theta.$ (e) $\rho = 5.$

(b) $\rho = 6 \sin \theta.$ (d) $\rho = -4 \sin \theta.$ (f) $\rho = -5.$

5. By changing the equation to rectangular coordinates show that $\rho = a (\cos \theta + \sin \theta)$ represents a circle. Draw the circle for $a = 4$.

6. Show that $\rho = a \cos \theta + b \sin \theta$ represents a circle.

7. Draw the circle $\rho = 3 \cos \theta + 2 \sin \theta$.

8. Change the following to polar coordinates.

(a) $x^2 + y^2 - 4x = 0.$ (d) $x^2 + y^2 - ax - ay = 0.$

(b) $x^2 + y^2 + 5y = 0.$ (e) $x^2 + y^2 - 8x - 8y + 16 = 0.$

(c) $x^2 + y^2 = a^2.$ (f) $x^2 + y^2 + ax - by = 0.$

9. Find the locus of the midpoints of all chords passing through a fixed point on a given circle.

10. A square with side 10 units in length has its center at the pole and one diagonal along the line $\theta = \pi/4$. Write the polar equations of the inscribed circle, the circumscribed circle, and the four circles which have as diameters the sides of the given square.

MISCELLANEOUS EXERCISES

1. Show that the x -axis intersects in two distinct points the circle $2x^2 + 2y^2 - 5x - y + 2 = 0$. Does the y -axis intersect the circle?

2. Show that the circle $x^2 + y^2 + 6x - 6y + 9 = 0$ is tangent to both axes.

3. Determine c so that the circle $2x^2 + 2y^2 - 8x + 5y + c = 0$ shall touch (a) the x -axis; (b) the y -axis.

4. Write the equation of the circle with center at the origin which touches the line $3x - 4y + 15 = 0$.

5. Write the equations of the two circles with centers at the origin which touch the circle $x^2 + y^2 - 8x - 20 = 0$.

6. Find the length of the chord which is cut from the line $4x - 3y - 2 = 0$ by the circle $x^2 + y^2 - 12x + 2y - 13 = 0$.

7. Write the equation of the circle which has for diameter the chord of the circle $x^2 + y^2 - x - 3y - 10 = 0$ which the latter cuts from the line $3x - 4y - 8 = 0$.

8. Find the equation of the circle which has for diameter the common chord of the circles $x^2 + y^2 + 4x - 4y - 12 = 0$ and $3x^2 + 3y^2 - 12x - 20y + 12 = 0$.

9. The midpoint of a chord of the circle $x^2 + y^2 + 5x - 25 = 0$ is $(\frac{3}{2}, 2)$. Find the length of the chord.

10. Find the midpoint of the chord which is cut from the line $x + 3y - 11 = 0$ by the circle $x^2 + y^2 - x + 3y - 30 = 0$.

11. Find the equation of the circle circumscribing the equilateral triangle whose vertices are $(-a, 0)$, $(a, 0)$, and $(0, a\sqrt{3})$.

12. Show that the quadrilateral whose vertices are $(-1, 3)$, $(1, 0)$, $(4, -2)$, and $(14, 13)$ can be circumscribed by a circle.

13. Write the equation of the circle tangent to both axes which has its center on the line $4x - 2y - 5 = 0$.

14. Write the equations of the two circles touching the lines $3x - 4y + 1 = 0$ and $4x + 3y - 7 = 0$ and passing through $(2, 3)$.

15. Write the equations of the two circles touching the line $3x + y - 6 = 0$ at $(1, 3)$ and having radii $2\sqrt{10}$.

16. Write the equation of the circle with minimum radius with center on the circle $x^2 + y^2 + 10x + 4y - 43 = 0$ and touching the line $x + y - 9 = 0$.

17. Write the equation of the circle with minimum diameter touching the circle $x^2 + y^2 + 4x + 12y - 50 = 0$ and the line $x + 3y = 30$.

18. A line segment $2\sqrt{5}$ units in length lies along the line $x - 2y + 4 = 0$. Find the coördinates of its end points if its mid-point is $(4, 4)$.

19. Find the points on the line $x + 2y - 9 = 0$ each at a distance of 5 units from $(2, 1)$.

20. Write the equation of the line tangent to the circle $x^2 + y^2 = 20$ at the point $(2, 4)$.

21. Show that $2x - 3y + 8 = 0$ is tangent to the circle $x^2 + y^2 - 2x + 2y - 11 = 0$ and find the equation of the parallel tangent.

22. Tangents to the circle $4x^2 + 4y^2 + 16x - 12y - 55 = 0$ are drawn at the points where it crosses the y -axis. Find the point of intersection of the tangents and their angle of intersection.

23. Tangents are drawn from the point $(6, -10)$ to the circle $x^2 + y^2 - 12x - 20y = 64$. Find the points where they touch the circle.

24. Show that the radical axes of any three circles, the circles being taken in pairs, either intersect in a common point or are parallel.

25. Find the point of intersection of the three radical axes of the circles which have for diameters the sides of the triangle whose vertices are $(-1, 2)$, $(5, 4)$, and $(3, -2)$.

26. Show that the centers of $x^2 + y^2 - 12x - 6y + 3 = 0$, $2x^2 + 2y^2 + 6x + 4y - 5 = 0$, and $3x^2 + 3y^2 - 6x - 2y + 1 = 0$ lie on a straight line.

27. Write the polar equations of the circles which have for diameters the sides of the triangle whose vertices are $(0, 0)$, $(8, 0)$, and $(0, 6)$.

28. Show that the perpendicular dropped from any point of a circle upon a diameter is a mean proportional between the segments into which it divides the diameter.

29. The base of a triangle is fixed in length and position. Find the locus of the vertex if the vertical angle is 90° .

30. Show that the locus of Exercise 21, page 69, is a circle.

31. A point moves so that the square of its distance from a given point equals its distance from a given line. Show that its locus is a circle.

32. Find the locus of a point if its distances from two fixed points are in a constant ratio k ($k \neq 1$).

33. Show that the locus of a point, the sum of the squares of whose distances from n fixed points is constant, is a circle.

34. A point moves so that the sum of the squares of its distances from two vertices of an equilateral triangle equals the square of its distance from the third vertex. Show that the locus is a circle and determine its position relative to the triangle.

35. The vertices of a triangle are $A(5, 10)$, $B(7, 4)$, and $C(-9, -4)$. The medians AD , BE , and CF intersect at G . The altitudes AK , BL , and CM intersect at H . The midpoints of AH , BH , and CH are R , S , and T , respectively. Show that the nine points D , E , F , K , L , M , R , S , and T lie on a circle whose center is on the line joining G and H and divides GH in the ratio $1:3$.

CHAPTER V

OTHER SECOND DEGREE CURVES

THE PARABOLA

57. Definition and equation of a parabola. A parabola is the locus of points which are equidistant from a fixed point and a fixed straight line. The fixed point is called the focus and the fixed line is called the directrix of the parabola.

In Fig. 55 (*a* and *b*) let AB be the directrix, and let F be the focus. Draw a line through F perpendicular to AB and

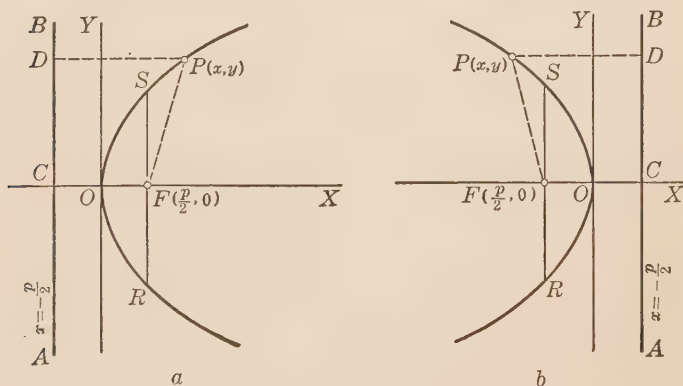


FIG. 55

intersecting it in C . By definition, the midpoint O of CF is a point on the parabola. Choosing O as origin and the line through C and F as the x -axis, we denote the coördinates of F by $(p/2, 0)$, where p is the distance from the directrix to the focus and is a positive number in Fig. 55 *a* and a negative number in Fig. 55 *b*. The equation of the directrix in each

case is $x = -p/2$. Let $P(x, y)$ be any point on the parabola and draw FP and DP . By definition,

$$FP = DP \text{ in Fig. 55 } a$$

and $FP = PD = -DP$ in Fig. 55 *b*.

$$\text{But } FP = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}, \text{ and } DP = x + \frac{p}{2};$$

$$\left. \begin{array}{l} \text{hence } \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = x + \frac{p}{2} \text{ in Fig. 55 } a \\ \text{and } \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = -\left(x + \frac{p}{2}\right) \text{ in Fig. 55 } b. \end{array} \right\} \quad (1)$$

Either form of equation (1) reduces to

$$x^2 - px + \frac{p^2}{4} + y^2 = x^2 + px + \frac{p^2}{4}. \quad (2)$$

Simplifying, we have

$$y^2 = 2px. \quad (3)$$

The simplicity of this equation is due to the particular way in which the axes have been chosen.

Conversely, the equation $y^2 = 2px$ represents a parabola.

Adding $x^2 - px + p^2/4$ to each member of equation (3), we have equation (2). Extracting the square root of each member of equation (2), we obtain the two equations (1). But equations (1) are satisfied by the coördinates of points which are equidistant from a point and a line, and by no other points. Hence equation (3) represents a parabola.

In case the focus of a parabola is the point $(0, p/2)$ and the directrix is the line $y = -p/2$, the equation of the parabola is

$$x^2 = 2py, \quad (4)$$

where p is positive if the focus is above the directrix and is negative if the focus is below the directrix.

It will be observed that p is the distance from directrix to focus.

58. Discussion of the equation. Solving equation (3) for y , we have

$$y = \pm \sqrt{2px}.$$

If p is positive (as shown in Fig. 55 *a*), x must be positive in order that y may be real. Hence the curve lies to the right of the y -axis. If p is negative, x must be negative, and the curve lies to the left of the y -axis (Fig. 55 *b*). To each value of x correspond two values of y , numerically equal but opposite in sign; hence the curve is symmetric with respect to the x -axis. As x increases numerically, y increases numerically and the curve extends indefinitely away from both axes. The line of symmetry is called the axis of the parabola, and the point of intersection of the parabola and its axis is called the **vertex** of the parabola. The double ordinate RS through the focus is called the **latus rectum**. Fig. 55 *a* is constructed for $p = 2$ and Fig. 55 *b* for $p = -2$.

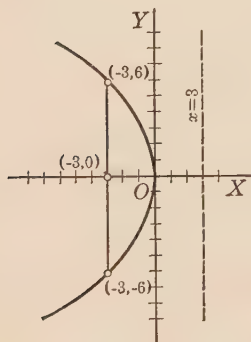


FIG. 56

EXAMPLE. Discuss and graph the parabola $y^2 = -12x$.

There is symmetry with respect to the x -axis. The parabola passes through the origin and does not cut either axis elsewhere. The curve lies to the left of the y -axis since x must be negative to give real values for y . The vertex of the parabola is at the origin. The distance from the directrix to the focus is $p = -6$; hence the focus is at $(-3, 0)$ and the equation of the directrix is $x = 3$. The length of the latus rectum is 12. The graph is shown in the figure.

EXERCISES

1. Find the coordinates of the focus and the equation of the directrix of each of the following parabolas. Sketch each curve.

(a) $y^2 = 4x$.

(c) $x^2 = 12y$.

(e) $2y^2 = 5x$.

(b) $y^2 = -6x$.

(d) $x^2 = -3y$.

(f) $5x^2 = 2y$.

2. Find the equations of the two parabolas through each of the following points, one with axis along the x -axis and the other with axis along the y -axis, the vertex of each parabola being at the origin: (a) $(3, 1)$; (b) $(5, -3)$; (c) $(-10, 4)$; (d) $(-\frac{1}{2}, -\frac{2}{3})$; (e) $(6, 6)$.

3. Draw on the same set of axes the parabolas $y^2 = 2px$ for $p = \frac{1}{2}, 1, 2, 3, 4$. What effect does the changing of the value of p have upon the shape of the parabola?

4. Derive the equation of a parabola with focus at $(p, 0)$ and with the y -axis as directrix.

5. Using the definition, write the equations of the following parabolas:

- (a) Focus at $(0, 0)$, directrix $x = 4$.
- (b) Focus at $(0, 0)$, directrix $x = -4$.
- (c) Focus at $(1, -3)$, directrix $y = 1$.
- (d) Focus at $(1, -\frac{3}{4})$, directrix $y = -\frac{5}{4}$.

6. Find the length of the latus rectum of $y^2 = 2px$.

7. Find the equation of the circle which has for diameter the double ordinate through the focus of $y^2 = 12x$. Show that the circle touches the directrix.

8. Show that the locus of the center of the circle which is tangent to a given line and which passes through a fixed point is a parabola.

9. An arch in the form of an arc of a parabola with its vertex at the center of the arch is 18 feet across at the base and its highest point is 8 feet above the base. What is the length of a beam parallel to the base and 6 feet above it?

10. A line AB is drawn parallel to the directrix of a parabola and intersecting the axis of the parabola in C . A circle with radius equal to the distance from the directrix to C is drawn with center at the focus. Show that the points of intersection of the circle with AB are on the parabola. How many additional points on the parabola be constructed? In this manner construct approximately a parabola if the distance from the directrix to the focus is 2 inches.

59. Vertex not at the origin. It is frequently necessary or convenient to study the parabola when the vertex is not at the origin. We shall obtain the equation when the vertex of

the parabola is at (h, k) and the axis of the curve is a line parallel to the x -axis.

Draw through the point (h, k) lines parallel to the x - and y -axes. If these lines are used as x' - and y' -axes, the vertex of the parabola will be at the origin and its equation will be

$$y'^2 = 2px'.$$

But $x = x' + h$ and $y = y' + k$,
or $x' = x - h$ and $y' = y - k$.
Making these substitutions,
we obtain

$$(y - k)^2 = 2p(x - h). \quad (5)$$

This is, therefore, the equation of a parabola with vertex at (h, k) and with axis parallel to the x -axis. Similarly, for a parabola with vertex at (h, k) and with axis parallel to the y -axis, the equation is

$$(x - h)^2 = 2p(y - k). \quad (6)$$

It will be noted that equations (5) and (6) become equations (3) and (4), respectively, for $h = k = 0$.

The translation of axes leaves unchanged the position of the focus and directrix with respect to the parabola.

Any equation of the form (5) or (6) can be put in the form of (3) or (4), respectively, by a translation of axes, and hence such an equation represents a parabola.

60. Theorem. *In general,* the equations*

$$Ay^2 + Bx + Cy + D = 0 \quad (7)$$

$$\text{and} \quad A'x^2 + B'y + C'x + D' = 0 \quad (8)$$

represent parabolas.

By completing squares and rearranging terms, the equations (7) and (8) can be put in the form of (5) and (6), respectively, and hence represent parabolas.

*Exceptions occur if the constants are such that the equations reduce to the first degree, or if the left members factor into first degree factors (see Sec. 40).

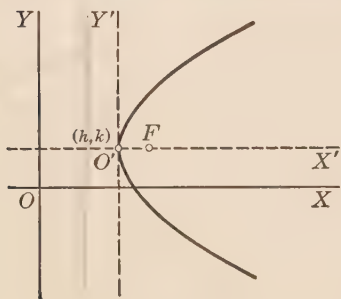


FIG. 57

EXAMPLE. Discuss and graph $y^2 - 6x - 8y - 2 = 0$.

Transposing and completing the square, we may write

$$y^2 - 8y + 16 = 6x + 18,$$

or

$$(y - 4)^2 = 6(x + 3).$$

The parabola represented by this equation has its vertex at $(-3, 4)$ and has for axis the line $y = 4$. The distance from directrix to focus is given by $2p = 6$, or $p = 3$. Since p is positive, the parabola opens toward the right. The focus is $\frac{3}{2}$ units to the right of the vertex and hence is $(-\frac{3}{2}, 4)$. The directrix is $x = -\frac{9}{2}$. The graph is shown in Fig. 58.

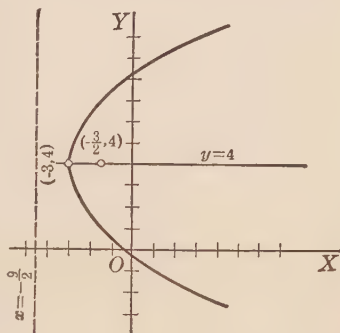


FIG. 58

61. Scientific applications of the parabola. In the following illustrations either an arc of a parabola is used or the variable quantities are related in the

same manner as the coördinates of a point on a parabola. The list is meant to be suggestive of the many applications of the parabola.

(a) The cable of a suspension bridge hangs in the form of an arc of a parabola.

(b) Some of the largest steel bridges are built with arches which are arcs of parabolas.

(c) A reflecting surface made by rotating a parabola about its axis will send the light out in parallel rays if the source of light is at the focus.

(d) The large reflector in a reflecting telescope is usually parabolic.

(e) The path of a projectile, if air resistance is neglected, is a parabola.

(f) The equation expressing the relation between the distance a freely falling body traverses and the time required is the equation of a parabola.

(g) The equation expressing the relation between the period and the length of a simple pendulum is the equation of a parabola.

(h) The equation expressing the relation between the bending moment at a point on a uniformly loaded beam and the distance from the point of support is the equation of a parabola.

(i) If a cylindrical vessel partly filled with water is whirled about the axis of the cylinder, a plane through the axis will cut the surface of the water in a parabola.

EXERCISES

1. Write the equations of the following parabolas and sketch :

(a) Directrix $x + 2 = 0$, vertex at $(1, 3)$.

(b) Directrix $y = 3$, vertex at $(-2, 2)$.

(c) Directrix $y = 0$, focus at $(3, 1)$.

(d) Directrix $x = 5$, focus at $(-1, 0)$.

(e) Vertex at $(-\frac{5}{2}, 1)$, focus at $(0, 1)$.

2. Determine the vertex, focus, length of latus rectum, directrix, and axis of each of the following and sketch the curve :

(a) $y^2 - 2y - 8x + 25 = 0$. (d) $2x^2 - 10x + 5y = 0$.

(b) $y^2 + 6y + 6x = 0$. (e) $3y^2 = 8x - 16$.

(c) $x^2 - 4x - 2y - 8 = 0$. (f) $5x^2 + 4y = 12$.

3. Find the length of the common chord of the parabolas $y^2 = 2x + 12$ and $y^2 + 2y = 4x$. Draw the figure.

4. Find the equation of the parabola with vertex at $(2, 1)$ and passing through $(5, -2)$ with axis (a) parallel to the x -axis; (b) parallel to the y -axis.

5. Show that $y = ax^2 + bx + c$ is a parabola, concave upward if $a > 0$ and concave downward if $a < 0$, and discuss the position of the curve if (a) $c = 0$; (b) $b = 0$; (c) $b = c = 0$.

6. Discuss the curve $x = ay^2 + by + c$.

7. Sketch the following curves :

(a) $y = 2x^2 - 5x - 3$. (d) $2y = x^2 - 2x + 5$.

(b) $y = 5 + 7x - 6x^2$. (e) $y = 4x^2 - x$.

(c) $x = 3y^2 + 11y - 4$. (f) $y = -(2 + x^2)$.

8. Find the equation of the parabola passing through $(1, -2)$, $(2, 1)$, and $(-1, 4)$ with axis (a) parallel to the y -axis; (b) parallel to the x -axis.

9. Show that the loci in Exercises 6 and 7, page 61, are parabolas.

10. Find the equation of the parabola which has for directrix the line $x + 2y = 1$ and whose focus is at the point $(1, 2)$.

THE ELLIPSE

62. Definition and equation of an ellipse. *An ellipse is the locus of points the sum of whose distances from two fixed points is constant.* The two fixed points are called the foci of the ellipse.

Let the distance between the fixed points be $2c$, and let the constant sum be $2a$, ($a > c$). If we choose the line through the foci as the x -axis and the point midway between the foci as origin, the coordinates of the foci are $(c, 0)$ and $(-c, 0)$. By definition

$$F'P + FP = 2a,$$

or
$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

Transposing the second radical, squaring, and reducing, we have

$$cx - a^2 = -a\sqrt{(x-c)^2 + y^2}.$$

Squaring again and reducing, we may write

$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2).$$

Since $a > c$, let $b^2 = a^2 - c^2$. Making this substitution and dividing by a^2b^2 , we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (9)$$

If the foci are the points $(0, c)$ and $(0, -c)$, the equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1. \quad (10)$$

Conversely, any equation of the form (9) or (10) represents an ellipse. The proof will not be set down here. It consists in

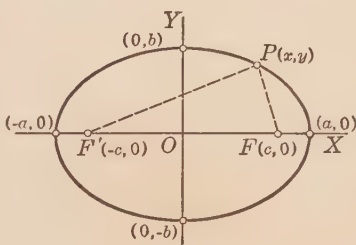


FIG. 59

showing that the steps used in obtaining equations (9) and (10) can be reversed, as was done in the case of the parabola.

63. Discussion of the equation. Equation (9) shows that the locus is symmetric with respect to both axes and hence is symmetric with respect to the origin. Solving for y , we have

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2};$$

whence it appears that real values of y exist only when $x^2 \leq a^2$. Hence the curve lies between the lines $x = a$ and $x = -a$. The largest numerical value is attained by y when $x = 0$; hence the curve lies between $y = b$ and $y = -b$. The x -intercepts are $\pm a$ and the y -intercepts are $\pm b$. The segment of the line of symmetry through the foci, from $(-a, 0)$ to $(a, 0)$, of length $2a$, is called the **major axis** of the ellipse. The segment of the line of symmetry at right angles to the major axis, from $(0, -b)$ to $(0, b)$, of length $2b$, is called the **minor axis**. The lengths a and b are the **semimajor** and **semiminor** axes, respectively. The point of intersection of the major and minor axes is the **center** since the ellipse is symmetric with respect to this point. The ends of the major axis are the **vertices** of the ellipse.

A similar discussion of equation (10) will show that it represents the same curve with major axis along the y -axis.

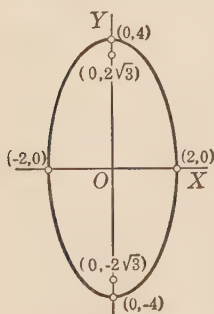


FIG. 60

EXAMPLE. Graph the ellipse $4x^2 + y^2 = 16$ and find its semiaxes and foci.

The equation shows symmetry with respect to both axes. Dividing by 16, we obtain

$$\frac{x^2}{4} + \frac{y^2}{16} = 1.$$

The equation shows that the major axis lies along the y -axis. The semimajor axis is $a = 4$ and the semiminor axis is $b = 2$. The vertices of the ellipse are at $(0, \pm 4)$ and the ends of the minor axis are at

$(\pm 2, 0)$. From $b^2 = a^2 - c^2$, we have $c = 2\sqrt{3}$. Hence the foci are at $(0, \pm 2\sqrt{3})$. The ellipse is shown in the figure.

EXERCISES

1. Find the semiaxes and foci of the following ellipses and sketch:

$$(a) \frac{x^2}{36} + \frac{y^2}{16} = 1.$$

$$(d) 9x^2 + y^2 = 45.$$

$$(e) 2x^2 + y^2 = 100.$$

$$(b) \frac{x^2}{9} + \frac{y^2}{25} = 1.$$

$$(f) x^2 + 4y^2 = 1.$$

$$(g) 4x^2 + 5y^2 = 20.$$

$$(c) x^2 + 3y^2 = 12.$$

$$(h) 50x^2 + y^2 = 200.$$

2. Write the equations of the following ellipses and sketch the curves:

(a) Major axis 8, foci at $(\pm 2, 0)$.

(b) Minor axis 10, foci at $(0, \pm 3)$.

~~(c)~~ Major axis 18, foci at $(0, \pm 3\sqrt{6})$.

(d) Minor axis $2\sqrt{3}$, foci at $(\pm\sqrt{3}, 0)$.

3. Find the equation of an ellipse with center at the origin and foci on the x -axis if (a) a focus bisects the semimajor axis; (b) the distance between the foci equals the minor axis.

~~4.~~ Find the equation of the ellipse with center at the origin and axes on the coordinate axes which passes through $(1, 3)$ and $(4, 2)$.

5. Find the locus of the midpoints of the ordinates of the circle $x^2 + y^2 = 36$.

6. Find the length of the double ordinate (latus rectum) through a focus of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

7. Write the equation of the parabola with vertex at the origin which passes through the ends of the double ordinate through the right focus of the ellipse $x^2 + 5y^2 = 20$.

8. Find the locus of a point which moves so that the sum of its distances from $(0, \pm 8)$ is 20.

9. Find the locus of a point which moves so that the sum of its distances from $(1, 1)$ and $(5, 1)$ is 8.

10. A semielliptic arch is 40 feet long across the base and 15 feet high with the major axis horizontal. Find the distance from the level of the top to a point on the arch (a) 10 feet from the minor axis; (b) 15 feet from the minor axis.

11. Two upright pins are fixed at the foci of an ellipse, and over them is placed a loop of string equal in length to the major axis of the ellipse plus the distance between the foci. A pencil point is pressed against the string so as to keep it taut. Show that as the pencil point moves around the foci it traces the ellipse.

64. Center not at the origin. Let the center of the ellipse be at the point (h, k) , and let the major axis be parallel to the x -axis. Draw through (h, k) lines parallel to the x - and y -axes and call these lines the x' - and y' -axes. The equation of the ellipse referred to these axes is, by (9) (Sec. 62),

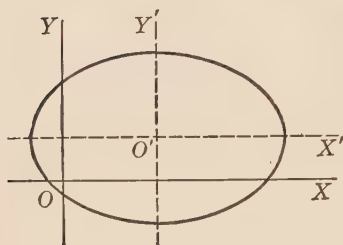


FIG. 61

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$$

But $x = x' + h$ and $y = y' + k$, or $x' = x - h$ and $y' = y - k$; and this gives

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1, \quad (11)$$

if the curve is referred to the x - and y -axes. If the major axis is parallel to the y -axis, the corresponding equation is

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1. \quad (12)$$

Since a translation of axes will change equations (11) and (12) into equations (9) and (10), respectively, any equation of the form of equation (11) or (12) represents an ellipse.

65. Eccentricity. Location of foci. The ratio c/a , denoted by e , is called the **eccentricity** of the ellipse. Since $c^2 = a^2 - b^2$, we may write

$$e = \frac{\sqrt{a^2 - b^2}}{a}.$$

This shows that as b approaches a , the eccentricity e approaches zero. But for $b = a$, the locus is a circle. Hence the circle is sometimes called an ellipse of eccentricity zero.

From the equation connecting a , b , and c , it is seen that a is the hypotenuse of a right triangle with legs b and c . Hence the distance from the end of the minor axis to a focus is a . The foci may, therefore, be located by drawing an arc with an end of the minor axis as center and a radius equal to a . The points in which the arc cuts the major axis are the foci.

66. Theorem. *The equation*

$$Ax^2 + By^2 + Cx + Dy + E = 0 \quad (13)$$

represents an ellipse if A and B are of the same sign.

If A and B are of the same sign, the equation may be written so that both are positive. Completing squares, we may write

$$A\left(x^2 + \frac{C}{A}x + \frac{C^2}{4A^2}\right) + B\left(y^2 + \frac{D}{B}y + \frac{D^2}{4B^2}\right) = \frac{C^2}{4A} + \frac{D^2}{4B} - E.$$

By dividing by the expression in the right member of the equation, we may write the equation in the form of equation (11) or (12) (Sec. 64). Equation (13), therefore, represents an ellipse.

If the quantity $C^2/4A + D^2/4B - E = 0$, the locus is a **point ellipse**.

If the quantity $C^2/4A + D^2/4B - E < 0$, there is no real locus.

EXAMPLE. Find the center, semiaxes, foci, and eccentricity of the ellipse $16x^2 + 25y^2 - 64x + 50y - 311 = 0$.

Completing squares, we may write

$$16(x^2 - 4x + 4) + 25(y^2 + 2y + 1) = 400.$$

Dividing by 400 and rearranging, we obtain

$$\frac{(x-2)^2}{25} + \frac{(y+1)^2}{16} = 1.$$

This equation shows that the center of the ellipse is at $(2, -1)$, that the major axis lies along the line $y = -1$, that the semimajor axis is $a = 5$, and that the semiminor axis is

$b = 4$. From $c^2 = a^2 - b^2$ we have $c = 3$. The foci are, therefore, at the points $(-1, -1)$ and $(5, -1)$. The eccentricity is $e = c/a = 3/5$.

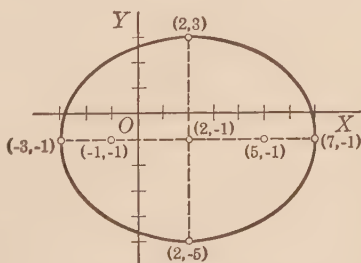


FIG. 62

67. A second definition of an ellipse. We shall now show that *an ellipse is the locus of a point which moves so that the ratio of its distance from a fixed point to its distance from a fixed line is a constant less than 1.*

Let the fixed line AB be the y -axis, and let the line through the fixed point F perpendicular to AB be the x -axis. If we call p the distance from AB to F , the coördinates of F are $(p, 0)$. By definition,

$$FP = e \cdot DP,$$

e being a constant less than 1.

But $FP = \sqrt{(x - p)^2 + y^2}$ and $DP = x$. Hence

$$\sqrt{(x - p)^2 + y^2} = ex,$$

or

$$x^2(1 - e^2) - 2px + y^2 + p^2 = 0.$$

This equation represents an ellipse (see Sec. 66) since $1 - e^2$ is positive ($e < 1$). The curve is symmetric with respect to the x -axis and is found to cut that axis in the points $M(p/(1 + e), 0)$ and $N(p/(1 - e), 0)$. The midpoint C of MN is the center of the ellipse and its coördinates are found to be $(p/(1 - e^2), 0)$. One semiaxis is MC , where

$$MC = OC - OM = \frac{p}{1 - e^2} - \frac{p}{1 + e} = \frac{pe}{1 - e^2}.$$

The other semiaxis is the ordinate CE and is found to be $pe/\sqrt{1 - e^2}$. It is easy to verify that MC is greater than CE and hence the major axis of the ellipse is MN .

In order to show that the point F is a focus of the ellipse as defined in Sec. 62, it is sufficient to show that

$$MC^2 - CE^2 = FC^2;$$

that is, that

$$\left(\frac{pe}{1 - e^2}\right)^2 - \left(\frac{pe}{\sqrt{1 - e^2}}\right)^2 = \left(\frac{p}{1 - e^2} - p\right)^2.$$

The equality is readily verified. Hence F is a focus of the ellipse.

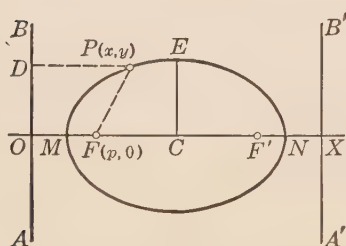


FIG. 63

a = dist. from centre to directrix

N.B.

The eccentricity of an ellipse was defined in Sec. 65 as the ratio c/a , where c represents the distance from the center to the focus of the ellipse and a is the semimajor axis. We shall show that the constant ratio e used in this section in the definition of the ellipse is the same as the eccentricity previously defined. The ratio

$$\frac{c}{a} = \frac{FC}{MC} = \frac{pe^2}{1-e^2} \div \frac{pe}{1-e^2} = e,$$

$$e = \frac{c}{a}$$

$$c = ae$$

a = dist. from center to directrix

and hence the ratio e and the eccentricity of the ellipse are identical.

The fixed line AB is called a **directrix** of the ellipse. The distance from the directrix to the center C is $p/(1-e^2)$. This is the semimajor axis $pe/(1-e^2)$ divided by e . Hence, if we call the semimajor axis a , the distance from the directrix to the center of the ellipse is a/e .

From the symmetry of the figure it is evident that there is another line $A'B'$ and another point F' which could serve as directrix and as focus, respectively, of the same ellipse.

68. Scientific applications of the ellipse. The following list is suggestive of the many applications of the ellipse:

(a) The planets move in elliptic orbits about the sun with the sun at a focus.

(b) The arches of stone and concrete bridges are frequently semi-ellipses.

(c) Elliptic gears are used in machines to obtain a slow, powerful movement with a quick return, as in power punches.

(d) Whispering galleries usually have elliptical ceilings arranged so that one may stand at a focus. Thus situated he can hear a slight noise made at the other focus, while an individual standing between the foci hears nothing.

(e) Because of its graceful beauty the elliptic arch is frequently used in architecture.

(f) The locus of the end of the radius vector which represents the magnetic field in a single-phase induction motor is an ellipse, assuming that the motor is operating at less than synchronous speed. This is the normal condition.

EXERCISES

1. Write the equations of the ellipses for which the following conditions are given and sketch each curve:

- ~~(a)~~ Major axis 10, foci at $(0, 2)$ and $(8, 2)$.
 (b) Major axis 12, foci at $(-1, 1)$ and $(-1, 5)$.
 (c) Minor axis 6, foci at $(1, -1)$ and $(7, -1)$.
 (d) Minor axis 8, foci at $(4, 3 - 2\sqrt{6})$ and $(4, 3 + 2\sqrt{6})$.

2. Find the center, semiaxes, and foci of the following ellipses and sketch each curve:

- ~~(a)~~ $x^2 + 2y^2 + 4x - 16y - 36 = 0$.
 (b) $3x^2 + y^2 + 12x + 6y - 3 = 0$.
 (c) $x^2 + 9y^2 - 12x - 36y + 36 = 0$.
 (d) $5x^2 + 4y^2 - 40x + 40y + 100 = 0$.
 (e) $4x^2 + 3y^2 + 8x - 12y = 0$.
 (f) $4x^2 + 25y^2 - 20x = 0$.
 (g) $2x^2 + y^2 - 10y - 25 = 0$.
 (h) $2x^2 + 3y^2 - 4x + 5y - 6 = 0$.

3. Find the eccentricity and the equations of the directrices of the following ellipses and draw figures showing each ellipse with its foci and directrices:

- ~~(a)~~ $16x^2 + 25y^2 = 400$.
 (b) $12x^2 + y^2 = 36$.
 (c) $5x^2 + 9y^2 - 30x = 0$.
 (d) $16x^2 + 20y^2 - 16x - 60y - 151 = 0$.

4. Write the equations of the ellipses for which the following conditions are given and sketch each figure:

- (a) $e = \frac{1}{2}$, directrices $x = \pm 12$, center at the origin.
 (b) $e = \frac{2}{3}$, directrices $x = \pm \frac{9\sqrt{5}}{2}$, center at the origin.
 (c) $e = \frac{3}{5}$, foci at $(2, -2)$ and $(2, 4)$.
 (d) Directrices $x = -6$ and $x = 10$, one focus at $(6, 3)$.

5. Show that the loci in Exercises 8 and 10, Sec. 45, are ellipses.

6. Write the equation of the ellipse which passes through $(-6, 4)$, $(-8, 1)$, $(2, -4)$, and $(8, -3)$ and which has axes parallel to the coordinate axes.

7. Show that the following equations represent point ellipses or no real loci:

- (a) $x^2 + 2y^2 = 0$.
 (b) $x^2 + 4y^2 - 2x - 16y + 17 = 0$.
 (c) $3x^2 + y^2 + 12x - 2y + 16 = 0$.

8. The foci of an ellipse are $(1, 1)$ and $(3, 3)$. The major axis is $4\sqrt{2}$. Write its equation, using the definition in Sec. 62.

9. The focus and corresponding directrix of an ellipse are $(1, 1)$ and $x + y + 4 = 0$, respectively. Its eccentricity is $\frac{1}{2}$. Write its equation, using the definition in Sec. 67.

THE HYPERBOLA

69. Definition and equation of a hyperbola. A hyperbola is the locus of points the difference of whose distances from two fixed points is a constant. The two fixed points are called the foci of the hyperbola.

Let the distance between the foci be $2c$, and let the constant difference be $2a$, ($a < c$). Choose the line through the foci as the x -axis and the point midway between the foci as origin. The coördinates of the foci are $(c, 0)$ and $(-c, 0)$. By definition, we have (Fig. 64)

$$F'P - FP = \pm 2a,$$

where the positive sign is used for a point P on the right of the y -axis and the negative sign is used for a point P on the left of the y -axis. But

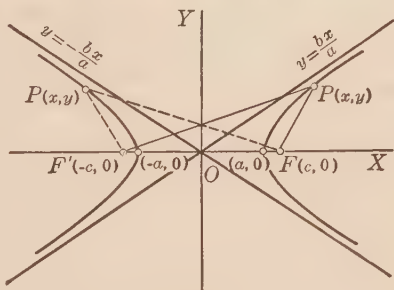


FIG. 64

$$F'P = \sqrt{(x+c)^2 + y^2} \quad \text{and} \quad FP = \sqrt{(x-c)^2 + y^2};$$

hence $\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$

Transposing the second radical, squaring, and reducing, we have

$$cx - a^2 = \pm a\sqrt{(x-c)^2 + y^2}.$$

Squaring again and reducing, we may write

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2).$$

Since $c > a$, let $b^2 = c^2 - a^2$. Making this substitution and dividing by a^2b^2 , we obtain the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad (14)$$

If the foci are the points $(0, c)$ and $(0, -c)$, the equation of the hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \quad (15)$$

Conversely, any equation of the form (14) or (15) represents a hyperbola. The proof consists in showing that the steps taken in obtaining (14) and (15) can be reversed. The work required involves only simple algebra and will not be set down.

70. Discussion of the equation. We shall now discuss equation (14). The equation shows that the locus is symmetric with respect to both axes and hence is symmetric with respect to the origin. The value of y ,

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2},$$

shows that x^2 cannot be less than a^2 ; that is, there is no part of the curve between the lines $x = \pm a$. The curve, therefore, does not cross the y -axis. The x -intercepts are $\pm a$. As x increases numerically, the numerical value of y increases. Write the value of y in the form

$$y = \pm \frac{b}{a} \sqrt{x^2 \left(1 - \frac{a^2}{x^2}\right)} = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}}.$$

The expression $1 - a^2/x^2$ approaches 1 as a limit as x increases without limit. Hence as x and y increase the curve approaches the two lines

$$y = \pm \frac{bx}{a}.$$

Since $\frac{bx}{a} \sqrt{1 - \frac{a^2}{x^2}}$ is numerically less than $\frac{bx}{a}$ for x equal to or numerically greater than a , the ordinate of the curve is numerically less than the corresponding ordinate of the line. The hyperbola represented by equation (14), therefore, consists of two parts or branches; one branch is to the right of $x = a$ and between the lines $y = \pm bx/a$, and the other branch is to the left of $x = -a$ and between the same lines. These lines $y = \pm bx/a$ are called the *asymptotes* of the hyperbola. The segment of the line of symmetry through the foci from

$(-a, 0)$ to $(a, 0)$, of length $2a$, is called the **transverse axis** of the hyperbola. The segment of the line of symmetry at right angles to the transverse axis, from $(0, -b)$ to $(0, b)$, of length $2b$, is called the **conjugate axis**. The lengths a and b are the **semitransverse** and **semiconjugate** axes, respectively. The point of intersection of the transverse and conjugate axes is the **center**, since the hyperbola is symmetric with respect to this point. The ends of the transverse axis are the **vertices** of the hyperbola.

A similar discussion of equation (15) will show that it represents the same curve with transverse axis along the y -axis.

EXAMPLE. Graph the hyperbola $9x^2 - 4y^2 = 36$ and find its foci, semiaxes, and asymptotes.

The equation shows symmetry with respect to both axes. Dividing by 36, we obtain

$$\frac{x^2}{4} - \frac{y^2}{9} = 1.$$

The equation shows that the transverse axis lies along the x -axis. The semitransverse axis is $a = 2$ and the semiconjugate axis is $b = 3$. The vertices of the hyperbola are at $(\pm 2, 0)$. From $b^2 = c^2 - a^2$ we have $c = \sqrt{13}$. Hence the foci are at $(\pm \sqrt{13}, 0)$. The asymptotes are $y = \pm 3x/2$. The hyperbola is shown in the figure.

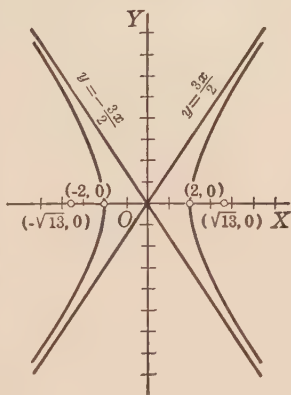


FIG. 65

EXERCISES

1. Find the semiaxes, foci, and equations of the asymptotes of the following hyperbolas and draw each curve and its asymptotes:

(a) $9x^2 - 16y^2 = 144$.

(e) $2x^2 - 3y^2 = 120$.

(b) $4y^2 - 5x^2 = 80$.

(f) $45y^2 - 4x^2 = 180$.

(c) $3x^2 - y^2 = 75$.

(g) $x^2 - y^2 = 16$.

(d) $x^2 - 9y^2 + 81 = 0$.

(h) $x^2 - y^2 = -16$.

2. Write the equations of the hyperbolas for which the following conditions are given and sketch each curve:

- (a) Transverse axis 8, foci at $(\pm 6, 0)$.
- ~~(b)~~ Transverse axis 6, foci at $(0, \pm 7)$.
- (c) Conjugate axis 4, foci at $(\pm 2\sqrt{5}, 0)$.
- (d) Conjugate axis $12\sqrt{2}$, foci at $(0, \pm 3\sqrt{10})$.

3. Find the equation of the hyperbola with center at the origin and transverse axis on the x -axis if the distance between the foci is double the distance between the vertices.

4. The asymptotes of a hyperbola are $3y = \pm 4x$ and the foci are at $(\pm 10, 0)$. Write the equation of the hyperbola.

5. Write the equation of the hyperbola passing through $(4, 6)$ whose asymptotes are $y = \pm \sqrt{3}x$.

6. A hyperbola with center at the origin and vertices on the x -axis passes through $(3, 1)$ and $(9, 5)$. Find its equation.

7. Find the length of the double ordinate (latus rectum) through a focus of the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

8. Write the equation of the ellipse passing through the origin which has for minor axis the double ordinate through the right focus of the hyperbola $x^2 - 3y^2 = 48$. Draw the figure.

9. Find the equation of the locus of a point which moves so that the difference of its distances from $(0, \pm 6)$ equals 4.

10. Find the equation of the locus of a point which moves so that the difference of its distances from $(-1, 3)$ and $(5, 3)$ is equal to 2.

11. A straightedge of length l has one of its ends pivoted at a point F_1 . A string has one end attached to the movable end of the straightedge and the other attached to a point F_2 . The string is of length $l - m$, where m is less than F_1F_2 . A pencil point is pressed against the string and the straightedge so as to keep the string taut. Show that as the straightedge turns about F_1 , the pencil point traces an arc of a hyperbola whose foci are F_1 and F_2 and whose transverse axis is m .

71. **Center not at the origin.** By an argument similar to that used for the ellipse one may show that for the center at (h, k) instead of at the origin the equation of a hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad (16)$$

if the transverse axis is parallel to the x -axis, and is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \quad (17)$$

if the transverse axis is parallel to the y -axis. Since a translation of axes will change equations (16) and (17) to equations (14) and (15), respectively, any equation of the form (16) or (17) represents a hyperbola.

72. Equilateral hyperbola. In case $a = b$, the hyperbola is called an equilateral hyperbola. Equation (14), in that case, reduces to

$$x^2 - y^2 = a^2.$$

The asymptotes have slopes ± 1 . If the axes are rotated through an angle of -45° , the equation becomes

$$x'y' = a^2. \quad (\text{See Exercise 9, p. 65})$$

This is the equation of an equilateral hyperbola referred to its asymptotes as axes.

73. Conjugate hyperbolas. Consideration of the two hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

shows that the transverse axis of one is the conjugate axis of the other, and that the same straight lines $y = \pm bx/a$ serve as asymptotes for both curves.

Two hyperbolas which are so related are called **conjugate hyperbolas**.

74. Eccentricity. Location of foci. The ratio c/a , denoted by e , is called the **eccentricity** of the hyperbola. Since $c^2 = a^2 + b^2$, we may write

$$e = \frac{\sqrt{a^2 + b^2}}{a}.$$

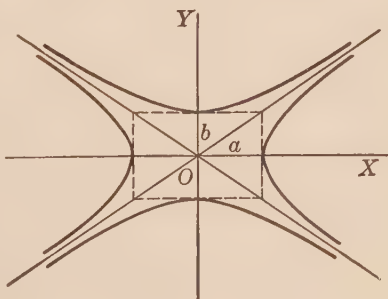


FIG. 66

From $c^2 = a^2 + b^2$ we see that c is the hypotenuse of a right triangle with legs a and b , and hence can be easily con-

structed if the semiaxes are known. The distance c laid off on either side of the center along the transverse axis locates the foci.

75. Theorem. *In general,* the equation*

$$Ax^2 + By^2 + Cx + Dy + E = 0 \quad (18)$$

represents a hyperbola if A and B are opposite in sign.

Completing squares, we may write

$$A\left(x^2 + \frac{C}{A}x + \frac{C^2}{4A^2}\right) + B\left(y^2 + \frac{D}{B}y + \frac{D^2}{4B^2}\right) = \frac{C^2}{4A} + \frac{D^2}{4B} - E.$$

Dividing by the expression in the right member of the equation and rearranging, we may write the equation in the form of (16) or (17), since A and B are opposite in sign. Equation (18), therefore, represents a hyperbola.

EXAMPLE. Find the center, semiaxes, asymptotes, foci, and eccentricity of the hyperbola

$$4y^2 - 9x^2 - 36x - 8y - 68 = 0.$$

Completing squares, we may write

$$4(y^2 - 2y + 1) - 9(x^2 + 4x + 4) = 36.$$

Dividing by 36 and rearranging, we obtain

$$\frac{(y-1)^2}{9} - \frac{(x+2)^2}{4} = 1.$$

This equation shows that the center of the hyperbola is at $(-2, 1)$, that the transverse axis lies along the line $x = -2$, that the semitransverse axis is $a = 3$, and that the semi-

conjugate axis is $b = 2$. From $c^2 = a^2 + b^2$ we have $c = \sqrt{13}$. The foci are, therefore, the points $(-2, 1 + \sqrt{13})$ and $(-2, 1 - \sqrt{13})$. The eccentricity is $e = c/a = \sqrt{13}/3$. The equations of the asymptotes are

$$y - 1 = \pm \frac{3}{2}(x + 2),$$

$$\text{or} \quad 3x - 2y + 8 = 0 \quad \text{and} \quad 3x + 2y + 4 = 0.$$

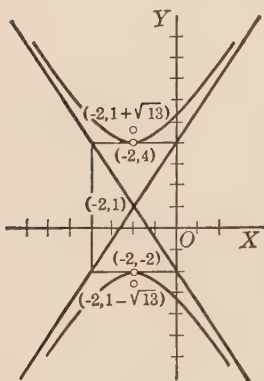


FIG. 67

*See footnote, p. 88.

76. A second definition of a hyperbola. One can show that *a hyperbola is the locus of a point which moves so that the ratio of its distance from a fixed point to its distance from a fixed line is a constant greater than 1*. The proof of this statement will not be given here since it follows step by step a similar proof for the case of the ellipse given in Sec. 67.

It can be shown that the fixed point is a focus and that the constant ratio is the eccentricity of the hyperbola as defined in Sec. 74.

The fixed line is called a **directrix** of the hyperbola and is perpendicular to the transverse axis. The distance from the directrix to the center of the hyperbola is a/e , where a is the semitransverse axis and e is the eccentricity. As in the case of the ellipse, there is a second directrix and a corresponding focus.

77. Scientific applications of the hyperbola. The following list is suggestive of the applications of the hyperbola:

(a) The equation expressing Boyle's law for a perfect gas under pressure at a constant temperature is of the form of the equation of an equilateral hyperbola referred to its asymptotes as axes.

(b) The hyperbola may be used to locate an invisible source of sound, as in locating an enemy's guns or in range-finding. Two listening posts are established and the difference in time of arrival of the report of the gun at the two posts is measured as accurately as possible. The difference in time multiplied by the rate at which sound travels gives the difference in the distances from the two listening posts to the gun. If the two listening posts are used as foci and the difference in distances from the gun as the transverse axis of a hyperbola, then the position of the gun will lie on the hyperbola. A second observation from new positions will give a second hyperbola through the position of the gun. The simultaneous solution of the equations of the two hyperbolas will locate the gun at one of the points of intersection of the two curves.

(c) A type of reflecting telescope uses a hyperbolic mirror as the small reflector to reflect the image to the eyepiece.

EXERCISES

1. Find the center, semiaxes, foci, and the equations of the asymptotes of the following hyperbolas and sketch each curve:

(a) $x^2 - 3y^2 - 4x + 18y - 50 = 0$.

(b) $16x^2 - 9y^2 + 96x + 72y + 144 = 0$.

(c) $4x^2 - 5y^2 + 8x + 40y + 4 = 0$.

(d) $3x^2 - y^2 - 18x - 6y = 0$.

(e) $40x^2 - 9y^2 + 126y - 81 = 0$.

(f) $x^2 - y^2 + 8x - 7y + 6 = 0$.

2. Find the eccentricity and the equations of the directrices of the following hyperbolas and in each case draw a figure showing the hyperbola, its foci, and its directrices:

(a) $7x^2 - 9y^2 = 63$.

(b) $x^2 - 8y^2 + 128 = 0$.

(c) $9x^2 - 16y^2 - 144x = 0$.

(d) $16x^2 - 81y^2 - 16x + 108y - 68 = 0$.

3. Find the equations of the hyperbolas for which the following conditions are given and sketch the figures:

(a) $e = \frac{3}{2}$, directrices $x = \pm \frac{4}{3}$, center at the origin.

(b) $e = \sqrt{2}$, directrices $x = \pm \sqrt{3}$, center at the origin.

(c) $e = 2$, foci at $(2, 4\sqrt{5})$ and $(2, -4\sqrt{5})$.

(d) Directrices $x = 2$ and $x = 6$, one focus at $(12, 0)$.

4. Show that the eccentricity of an equilateral hyperbola is $\sqrt{2}$.

5. Find the vertices, axes, foci, and directrices of the hyperbola $xy = 8$.

6. Write the equations of the hyperbolas conjugate to the following hyperbolas. Sketch each pair of conjugate hyperbolas on the same set of axes with their common asymptotes.

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

(c) $x^2 - y^2 = 10$.

(d) $2x^2 - y^2 = 12$.

(b) $\frac{x^2}{4} - \frac{y^2}{12} = -1$.

(e) $xy = 9$.

7. Show that the four foci of two conjugate hyperbolas and the four points of intersection of the tangent lines at the vertices all lie on a circle whose center is the common center of the hyperbolas.

8. Show that the loci in Exercises 9 and 11, page 61, are hyperbolas.

9. Write the equation of the hyperbola which passes through the points (8, 1), (0, -5), (6, -2), and (2, -2), and which has axes parallel to the coördinate axes.

10. A hyperbola has its foci at (2, -2) and (-6, 6). Its transverse axis is $4\sqrt{2}$. Find its equation, using the definition in Sec. 69.

11. A directrix and corresponding focus of a hyperbola are $x - y + 2 = 0$ and (2, -2), respectively. The eccentricity is 2. Find its equation, using the definition in Sec. 76.

CONICS

78. General second degree equation. The general equation of the second degree in x and y can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \quad (19)$$

It is possible by a proper rotation of axes to obtain an equation of the locus represented by equation (19) in a form which has no xy -term. Rotating the axes through the angle θ by the substitutions

$$\begin{aligned} x &= x' \cos \theta - y' \sin \theta, \\ y &= x' \sin \theta + y' \cos \theta, \end{aligned}$$

and collecting terms, we transform equation (19) into

$$\begin{aligned} &x'^2(A \cos^2 \theta + B \sin \theta \cos \theta + C \sin^2 \theta) \\ &+ x'y'[2(C - A) \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta)] \\ &+ y'^2(A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta) \\ &+ x'(D \cos \theta + E \sin \theta) + y'(E \cos \theta - D \sin \theta) \\ &+ F = 0. \end{aligned}$$

Equating the coefficient of $x'y'$ to zero, we have

$$2(C - A) \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) = 0,$$

$$\text{or} \quad (C - A) \sin 2\theta + B \cos 2\theta = 0,$$

$$\text{whence} \quad \tan 2\theta = \frac{B}{A - C}. \quad (A \neq C)$$

In case $A = C$, we have $\cos 2\theta = 0$ and hence $\theta = 45^\circ$. There is a value of θ for which $\tan 2\theta$ is equal to any given value. Therefore it is always possible to choose an angle

through which to rotate the axes, and eliminate the xy -term in the general second degree equation in x and y . Since the xy -term can always be removed, we shall have no loss of generality by discussing the equation in the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

1. If the left member of the equation can be factored into two first degree factors, as $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$, the equation represents two straight lines (Sec. 40). These lines may be distinct or coincident.

In case the equation does not represent straight lines, there are four possibilities.

2. If $A = C$, the equation represents a circle (Sec. 52).

3. If either $A = 0$ or $C = 0$, the equation represents a parabola (Sec. 60).

4. If A and C have the same sign and $A \neq C$, the equation represents an ellipse (Sec. 66).

5. If A and C are opposite in sign, the equation represents a hyperbola (Sec. 75).

Therefore every second degree equation of the form (19) represents one of the following loci: two straight lines, a circle, a parabola, an ellipse, or a hyperbola. These five types of curves are called **conics**, or **conic sections**, because each can be obtained as a curve of intersection of a plane and a conical surface. They were so studied by the early Greeks.

Equations of types 2 and 4 represent imaginary loci in some cases.

79. General definition of a conic. A conic is the locus of a point the ratio of whose distance from a fixed point to its distance from a fixed line is a constant. The fixed point is called the focus, the fixed line is called the directrix, and the constant ratio is called the eccentricity of the conic.

From Secs. 57, 67, and 76 we see that the conic is a parabola, an ellipse, or a hyperbola according as the eccentricity is $e = 1$, $e < 1$, or $e > 1$, respectively.

EXAMPLE 1. Identify the conic

$$11x^2 - 24xy + 4y^2 + 30x + 40y - 45 = 0.$$

The angle θ through which to rotate the axes into a position parallel to the axes of the conic is found from

$$\tan 2\theta = \frac{B}{A-C} = -\frac{24}{7}.$$

If we select for 2θ the smallest positive angle with the given tangent, we have $\cos 2\theta = -7/25$, from which

$$\sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \frac{4}{5}$$

and $\cos \theta = \frac{3}{5}.$

The equations for the required rotation are

$$x = \frac{3x' - 4y'}{5}, \quad y = \frac{4x' + 3y'}{5}.$$

These substitutions transform the equation into

$$\begin{aligned} 11\left(\frac{3x' - 4y'}{5}\right)^2 - 24\left(\frac{3x' - 4y'}{5}\right)\left(\frac{4x' + 3y'}{5}\right) \\ + 4\left(\frac{4x' + 3y'}{5}\right)^2 + 30\left(\frac{3x' - 4y'}{5}\right) \\ + 40\left(\frac{4x' + 3y'}{5}\right) - 45 = 0. \end{aligned}$$

Simplifying, we obtain

$$x'^2 - 4y'^2 - 10x' + 9 = 0.$$

This equation shows that the conic is a hyperbola. Completing squares and dividing by 16, we may write

$$\frac{(x' - 5)^2}{16} - \frac{y'^2}{4} = 1.$$

The transverse axis of the hyperbola lies on the x' -axis. The semitransverse axis is $a=4$ and the semiconjugate axis is $b=2$. With reference to the new axes the center is at $(5, 0)$ and the vertices are at $(1, 0)$ and $(9, 0)$. The value

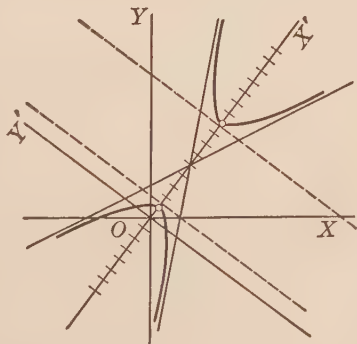


FIG. 68

of c is found to be $2\sqrt{5}$ and the eccentricity $e = \sqrt{5}/2$. Hence the foci are at $(5 \pm 2\sqrt{5}, 0)$ and the equations of the directrices are $x' = 5 \pm 8\sqrt{5}/5$. The asymptotes are $y' = \pm \frac{1}{2}(x' - 5)$. The figure shows the relation of the curve to both sets of axes.

A method of identifying a conic without the removal of the xy -term is illustrated in Example 2.

EXAMPLE 2. Identify the curve represented by

$$x^2 - xy + y^2 = 3.$$

Solving as a quadratic in y , we have

$$y = \frac{x}{2} \pm \frac{1}{2} \sqrt{3(4 - x^2)}.$$

This equation shows that for a value of x there are two values of y , giving points at equal vertical distances above and below

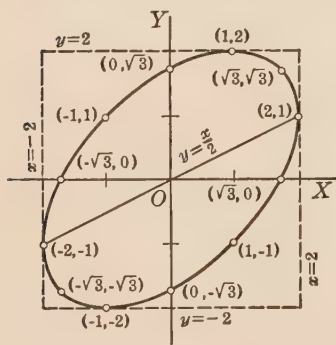


FIG. 69

the corresponding point on the line $y = x/2$. That is, the line $y = x/2$ bisects the system of chords drawn parallel to the y -axis. A study of the quantity under the radical shows that the curve lies between the lines $x = \pm 2$. For such values of x , the value of y is limited. Hence the equation represents an ellipse. It may be necessary to plot a few points, since the curve is not symmetric in the

ordinary sense to the line $y = x/2$. The curve is shown in the figure.

EXERCISES

1. Remove the xy -term by rotation of axes and identify the following conics:

- $9x^2 - 24xy + 16y^2 - 20x - 15y = 0$.
- $9x^2 + 24xy + 16y^2 - 20x + 15y - 100 = 0$.
- $41x^2 - 24xy + 34y^2 - 15x - 20y = 0$.
- $15x^2 + 6xy + 15y^2 - 3\sqrt{2}x - 3\sqrt{2}y - 34 = 0$.
- $x^2 + 2\sqrt{3}xy - y^2 = 18$.
- $xy + 3x - 4y = 8$.

2. Solve for x or for y , sketch, and identify the following conics :

(a) $5x^2 + 4xy + 4y^2 - 36 = 0$. (c) $3x^2 + 4xy + y^2 + 4 = 0$.

(b) $x^2 - 2xy + y^2 - x + 4 = 0$. (d) $x^2 + 2xy + y^2 - 2 = 0$.

3. Translate the origin to $(3, 4)$, then rotate the axes to remove the xy -term, and identify $11x^2 - 24xy + 4y^2 + 30x + 40y - 45 = 0$.

4. Transform the equation $x^2 + 2y^2 - x + 4y + 3 = 0$ by rotating the axes through an angle of 90° .

5. Show that each of the following equations represents two intersecting lines :

(a) $2x^2 + 3xy - 2y^2 - 5y - 2 = 0$.

(b) $x^2 - xy - 2y^2 + x + 4y - 2 = 0$.

(c) $xy + 2x - 3y - 6 = 0$.

(d) $xy = 0$.

6. Show that each of the following equations represents two parallel lines :

(a) $4x^2 + 4xy + y^2 + 4x + 2y - 3 = 0$. (c) $x^2 - 2x - 8 = 0$.

(b) $x^2 - 2xy + y^2 + x - y - 2 = 0$. (d) $y^2 = 4y$.

7. Show that each of the following equations represents two coincident lines :

(a) $x^2 + 4xy + 4y^2 - 2x - 4y + 1 = 0$. (c) $x^2 - 2xy + y^2 = 0$.

(b) $y^2 + 2y + 1 = 0$. (d) $x^2 = 0$.

Show that the locus in

8. Exercise 13, page 61, is a parabola.

9. Exercise 12, page 61, is an ellipse.

10. Exercise 15, page 61, is a hyperbola.

11. Find the equation of the conic through the points $(3, 0)$, $(0, -5)$, $(2, 1)$, $(2, -2)$, and $(-3, -2)$.

80. Polar equation of a conic. The polar equation of a conic can easily be obtained from the definition in the preceding section. Choose the focus for the pole and the line through the focus perpendicular to the directrix for the polar axis. By definition,

$$FP = eDP. \text{ (Fig. 70)}$$

But $FP = \rho$ and $DP = p + \rho \cos \theta$. Hence

$$\rho = ep + e\rho \cos \theta.$$

Solving for ρ , we have

$$\rho = \frac{ep}{1 - e \cos \theta}.$$

Figs. 71, 72, and 73 are drawn for $e = 1$, $e = \frac{1}{2}$, and $e = 2$, respectively, and represent a parabola, an ellipse, and a hyperbola.

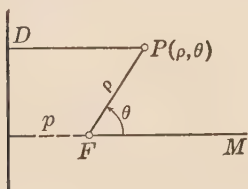


FIG. 70

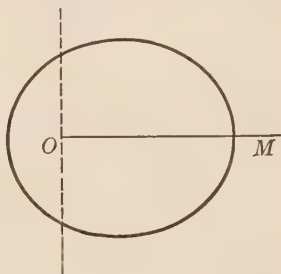


FIG. 72

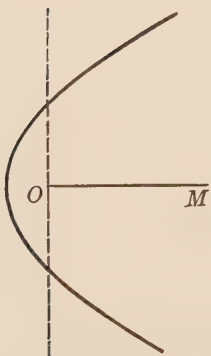


FIG. 71

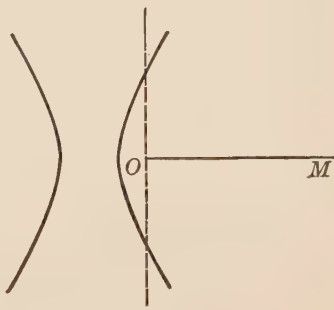


FIG. 73

EXERCISES

1. Graph the following equations:

$$(a) \rho = \frac{6}{1 - \cos \theta}.$$

$$(d) \rho = \frac{4}{2 + \sin \theta}.$$

$$(b) \rho = \frac{6}{1 - \sin \theta}.$$

$$(e) \rho = \frac{8}{1 + 3 \cos \theta}.$$

$$(c) \rho = \frac{4}{2 + \cos \theta}.$$

$$(f) \rho = \frac{8}{1 + 3 \sin \theta}.$$

2. Show that $\rho = a/(\cos \theta + 1)$ and $\rho = a/(\cos \theta - 1)$ represent the same parabola.

3. Show that $\rho = a \sec^2 \frac{\theta}{2}$ is a parabola, and sketch.
4. Change $y^2 = 2px$ to an equation in polar coördinates.
5. Change $x^2/a^2 + y^2/b^2 = 1$ to an equation in polar coördinates.
6. Change $x^2/a^2 - y^2/b^2 = 1$ to an equation in polar coördinates.
7. Why do the results in Exercises 4, 5, and 6 differ from the general polar equation of the conic developed in the text?
8. Change the general polar equation of the conic to an equation in rectangular coördinates. Show that by proper choice of e this reduces to typical forms of the rectangular equations of conics.

MISCELLANEOUS EXERCISES

1. Identify the following loci :

- | | |
|---------------------------------|---|
| (a) $2x^2 + 5y^2 = 10.$ | (h) $2x^2 + 2y^2 + 6x = 7.$ |
| (b) $x^2 - 4y = 8.$ | (i) $3x^2 - 5y^2 = 3x + 5y.$ |
| (c) $6x^2 - y^2 = 18.$ | (j) $7x^2 - y = x + 6.$ |
| (d) $5x^2 + 5y^2 = 3.$ | (k) $y^2 - 4 = 4x.$ |
| (e) $12y^2 + 8y = x.$ | (l) $y = \pm \frac{b}{a} \sqrt{2ax - x^2}.$ |
| (f) $3x^2 - y^2 = 0.$ | |
| (g) $x^2 + 6y^2 - 3x + 4y = 1.$ | |

2. Identify the following loci :

- (a) $2x^2 + 2xy + y^2 = 9.$
- (b) $x^2 + 6xy + y^2 = 4.$
- (c) $4x^2 - 4xy + y^2 = 4x.$
- (d) $x^2 + 2xy + y^2 + x + 2y = 0.$
- (e) $x^2 - xy + y^2 + x + y = 2.$
- (f) $(x - 2)(y - 1) = 3.$
- (g) $x^2 - 2xy + y^2 + 4x - 4y + 3 = 0.$
- (h) $xy = 3y.$

3. Identify the following loci :

- | | |
|------------------------------------|---|
| (a) $\rho(2 - \cos \theta) = 3.$ | (d) $\rho \sin \theta = 5.$ |
| (b) $\rho(1 - \cos \theta) = 2.$ | (e) $\rho = a \csc^2 \frac{1}{2} \theta.$ |
| (c) $\rho(2 + 3 \sin \theta) = 1.$ | (f) $\rho^2 = 2a^2 \csc 2\theta.$ |

4. Find the length of the common chord of $y^2 = 12x$ and $2x^2 = 3y.$

5. Find the vertices of the square inscribed in the ellipse $x^2 + 4y^2 = 8x.$

6. Find the points of intersection of $x^2 + xy + y^2 = 3$ and $x^2 + y^2 = 3$ and check graphically.

7. Find the points of intersection of $x^2 - xy + y^2 = 3$ and $x^2 + y^2 = 6$ and check graphically.

8. Find the points of intersection of $4x^2 + 4y^2 - 7x - 26y = 0$ and $2y = x^2 - 3x$ and check graphically.

9. Find the points of intersection of $3x^2 + 8y^2 - 24x = 0$ and $8x^2 + 5y^2 - 64x + 48 = 0$ and check graphically.

10. Find the points of intersection of the curves $xy = 1$ and $x^2 - 2xy + y^2 - 8x - 8y - 16 = 0$ and check graphically.

11. Find the equation of the circle through the vertex and the extremities of the double ordinate through the focus of $y^2 = 2px$.

12. An isosceles right triangle is inscribed in the parabola $y^2 = 2px$ so that the vertex of the right angle is at the origin. Find the length of the hypotenuse.

13. The vertex of a parabola is at the origin and its axis lies on the y -axis. Find the length of the chord through the focus if one end is at the point $(6, 6)$.

14. Any two perpendicular lines are drawn through the vertex of a parabola. Show that the line joining their other intersections with the parabola intersects the axis of the parabola in a fixed point.

15. The cable of a suspension bridge hangs in the shape of an arc of a parabola. The supporting towers are 70 feet high and 200 feet apart and the lowest point on the cable is 20 feet above the roadway. Find the length of a supporting rod 50 feet from the middle of the bridge.

16. The semi-axes of an ellipse are 3 feet 4 inches and 2 feet, respectively. Find the lengths of those diameters of the ellipse which make angles of 30° and 60° , respectively, with the major axis. (A diameter is a chord through the center.)

17. Chords are drawn through an end of the major axis of the ellipse $b^2x^2 + a^2y^2 = 2ab^2x$. Find the locus of their midpoints.

18. Write the equation of the ellipse the ends of whose minor axis are at the foci of the hyperbola $9x^2 - 16y^2 = -144$ and whose eccentricity is the reciprocal of the eccentricity of the hyperbola.

19. A rod 12 inches long has its ends moving on two perpendicular lines. (a) Find the locus of its midpoint. (b) Find the loci of its points of trisection.

20. Lines are drawn through the ends of the minor axis of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ intersecting in a point on the ellipse. Show that the product of their intercepts on the x -axis is a^2 .

21. The base of a triangle is fixed in length and position. Find the locus of its vertex if the tangents of the base angles have (a) a constant sum; (b) a constant difference; (c) a constant product; (d) a constant quotient.

22. Show that $(x - a)(y - b) = c$ represents an equilateral hyperbola. What are its asymptotes?

23. Show that $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ represents an arc of a parabola. Find its vertex, focus, and directrix. (Rationalize the equation.)

24. Two fixed lines intersect at an angle $\tan^{-1}k$. Show that the locus of a point, the product of whose distances from the fixed lines is constant, is a hyperbola.

25. If 2ϕ is the angle between the asymptotes of a hyperbola, show that $\sec \phi = e$, where e is the eccentricity.

26. If e and e' are the eccentricities of conjugate hyperbolas, show that $e^2 + e'^2 = e^2e'^2$.

27. Show that the segment of an asymptote of a hyperbola included between the directrices equals the transverse axis.

28. Show that the distance from the center to a point on an equilateral hyperbola is a mean proportional between the focal distances of the point.

29. The line segment between the center O and the focus F of a hyperbola is the diameter of a circle which intersects an asymptote in P . Show that the chords OP and FP are equal to the semi-transverse and semiconjugate axes, respectively.

30. Let P be any point on a hyperbola. Show that the product of the distances from the asymptotes to P is constant.

31. If a line is drawn intersecting a hyperbola, show that the two segments of the line included between the hyperbola and its asymptotes are equal.

32. Lines are drawn through the vertices of an equilateral hyperbola intersecting in a point on the hyperbola. Show that the bisector of the angle formed by these lines is parallel to an asymptote.

33. Derive the polar equation of a parabola if the pole is at the vertex.

34. Derive the polar equation of an ellipse if the pole is at the center.

35. Sketch (a) $\rho = 2 \cos \theta / \cos 2 \theta$; (b) $\rho = 2 \sin \theta / \sin 2 \theta$. Identify the loci.

36. Show that the vertices of the convex hexagon, $(-\frac{1}{4}, 0)$, $(0, 1)$, $(6, 5)$, $(12, 7)$, $(6, -5)$, and $(2, -3)$, lie on a parabola with axis along the x -axis. Show that the pairs of opposite sides of the hexagon meet in three points which lie on a straight line.

CHAPTER VI

TANGENTS, NORMALS, DIAMETERS, POLES, AND POLARS

81. Secants and tangents. A line which cuts across a curve is called a **secant** of the curve. The line P_1P_2 in Fig. 74 or Fig. 75 is such a line.

If the point P_1 of intersection of a secant with a curve (Fig. 74 or Fig. 75) is held fixed and the point P_2 of intersection moves along the curve until it coincides with P_1 , then the line P_1P_2 approaches a limiting position P_1T . The line P_1T formed in this way is called a **tangent** to the curve at the point P_1 , and P_1 is called the **point of contact** of the tangent line.

The **tangent** to a curve at a point P_1 on the curve is the limiting position of the secant line P_1P_2 when P_1 is held fixed and P_2 moves along the curve until it coincides with P_1 .

The **normal** to a curve at the point P_1 on the curve is the line which is perpendicular at P_1 to the tangent at P_1 .

If (x_1, y_1) is the point P_1 on a curve, then (see Sec. 16) the equation of the tangent at P_1 is

$$y - y_1 = m(x - x_1), \quad (1)$$

where m is the slope of the tangent at P_1 . Therefore when the slope of the tangent line at P_1 is determined we can write down at once the equation of the tangent to the curve at P_1 . We can easily write the equation of the normal at P_1 since it is perpendicular to the tangent.

By means of examples we will now illustrate the method of finding the slope of the tangent line to a curve at a given point.

EXAMPLE 1. Find the equation of the tangent to the parabola $y^2 = 2px$ at the point $P_1(x_1, y_1)$ on the parabola.

Denote the point $P_2(x_2, y_2)$ on the parabola by $(x_1 + h, y_1 + k)$, so that $x_2 = x_1 + h$ and $y_2 = y_1 + k$. Then the slope m of P_1P_2 is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{k}{h}.$$

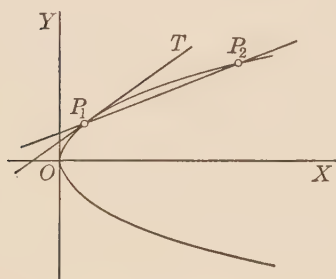


FIG. 74

We have to find the value which k/h approaches as k and h both approach zero. Since P_1 and P_2 are on the parabola, we have

$$y_1^2 = 2px_1, \quad (y_1 + k)^2 = 2p(x_1 + h) \quad \text{or} \quad y_1^2 + 2y_1k + k^2 = 2px_1 + 2ph.$$

Subtracting the first of these equations member by member from the second, we have

$$2y_1k + k^2 = 2ph.$$

From this it follows that

$$\frac{k}{h} = \frac{2p}{2y_1 + k}.$$

Then as h and k both approach zero the second member of this equation approaches p/y_1 . Hence the limiting value of k/h is p/y_1 . But this limiting value is the slope of the tangent P_1T . Hence the slope of P_1T is p/y_1 . Substituting this value for m in equation (1), we have

$$y - y_1 = \frac{p}{y_1} (x - x_1).$$

On clearing of fractions, we have

$$y_1y - y_1^2 = px - px_1.$$

Substituting for y_1^2 its value $2px_1$ and simplifying, we have

$$y_1y = p(x + x_1).$$

Therefore the equation of the tangent line to the parabola $y^2 = 2px$ at the point $P_1(x_1, y_1)$ on the parabola is

$$y_1y = p(x + x_1).$$

EXAMPLE 2. Find the equation of the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point $P_1(x_1, y_1)$ on the ellipse.

Denoting $P_2(x_2, y_2)$ by $(x_1 + h, y_1 + k)$, as in the preceding example, we have

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{k}{h}.$$

Hence we have again to find the limiting value of k/h .

Since P_1 and P_2 are on the ellipse, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1,$$

$$\frac{x_1^2 + 2x_1h + h^2}{a^2} + \frac{y_1^2 + 2y_1k + k^2}{b^2} = 1.$$

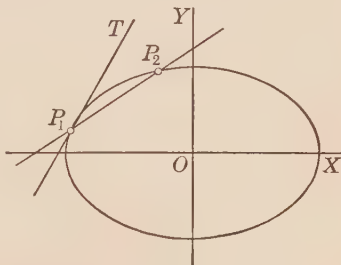


FIG. 75

Subtracting the first of these equations member by member from the second, we have

$$\frac{2x_1h + h^2}{a^2} + \frac{2y_1k + k^2}{b^2} = 0;$$

whence it follows that

$$a^2k(2y_1 + k) = -b^2h(2x_1 + h).$$

Hence
$$\frac{k}{h} = -\frac{b^2(2x_1 + h)}{a^2(2y_1 + k)}.$$

When h and k both approach zero the second member of this equation approaches $-b^2x_1/a^2y_1$. This is the limiting value of k/h ; it is therefore the slope of the tangent at P_1 . Putting this value for m in equation (1) we have for the tangent P_1T the equation

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1).$$

On clearing of fractions and parentheses, we have

$$a^2y_1y - a^2y_1^2 = -b^2x_1x + b^2x_1^2;$$

or
$$b^2x_1x + a^2y_1y = b^2x_1^2 + a^2y_1^2.$$

But the second member has the value a^2b^2 , since (x_1, y_1) is on the given ellipse; hence

$$b^2x_1x + a^2y_1y = a^2b^2,$$

or
$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

Therefore *the equation of the tangent line to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point $P_1(x_1, y_1)$ on the ellipse is

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

In a similar way the student may prove the following :

The equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point $P_1(x_1, y_1)$ on the hyperbola is

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1.$$

The foregoing process of finding the slope of the tangent may be described as follows :

To find the slope of the tangent to a given curve at the point $P_1(x_1, y_1)$ on the curve, take a second point $P_2(x_1 + h, y_1 + k)$ on the curve and observe that k/h is the slope of the secant P_1P_2 ; then substitute the coördinates of P_1 and P_2 in the equation of the given curve and subtract one of the resulting equations from the other member by member, and from the new equation thus obtained find a convenient value for k/h ; then find the limiting value of k/h as k and h both approach zero: this is the required slope of the tangent line.

Since the normal to a curve at $P_1(x_1, y_1)$ is perpendicular to the tangent to the curve at this point it follows (see Sec. 9) that the slope of the normal is the negative reciprocal of the

slope m of the tangent at this point; that is, the slope of the normal is $-1/m$. Since the normal passes through (x_1, y_1) it follows therefore that its equation is

$$y - y_1 = -\frac{1}{m}(x - x_1).$$

EXAMPLE 3. Find the equation of the normal to the ellipse in Example 2 at the point $P_1(x_1, y_1)$.

Since the slope of the tangent was found to be $-b^2x_1/a^2y_1$ it follows that the slope of the normal is a^2y_1/b^2x_1 . Hence the equation of the normal is

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1).$$

EXERCISES

1. Derive the equation of the tangent to each of the following curves at the point $P_1(x_1, y_1)$ on the curve:

~~(a)~~ $xy = c$.

(b) $x^2 + y^2 = r^2$.

(c) $y = x^3$.

(d) $y^2 = x^3$.

(e) $y = ax^2 + bx + c$.

~~(f)~~ $y^2 = ax + b$.

2. Find the equations of the tangents and normals to the following curves at the indicated points. In each case draw the curve and its tangent and normal at the given point.

~~(a)~~ $y^2 = x^3$ at $(1, 1)$.

(b) $x^2 + y^2 = 25$ at $(3, 4)$.

~~(c)~~ $y = \frac{x+1}{x-1}$ at $(2, 3)$.

(d) $xy = 4$ at $(1, 4)$ and also at $(4, 1)$.

(e) $y = x^2 - 4x + 2$ at $(4, 2)$.

3. Show that the normal to a circle at any point passes through the center of the circle.

4. Prove that the area inclosed by the coordinate axes and any tangent to the hyperbola $2xy = a^2$ is equal to a^2 .

5. Find the points P on the hyperbola $2xy = a^2$ such that the normal through P shall pass through the origin of coordinates.

6. For what points of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ do the normals pass through the center of the ellipse?

82. **Tangent to a general curve of the second degree.** Let us write the general equation of the second degree in two variables in the form

$$Ax^2 + 2 Bxy + Cy^2 + 2 Dx + 2 Ey + F = 0. \quad (2)$$

Let $P_1(x_1, y_1)$ be a point on the conic defined by this equation. Employing the method of the preceding section, we shall now find the slope of the tangent to this curve at the point P_1 . Let $P_2(x_1 + h, y_1 + k)$ be a second point on the curve. Since P_1 and P_2 are on the curve, we have

$$Ax_1^2 + 2 Bx_1y_1 + Cy_1^2 + 2 Dx_1 + 2 Ey_1 + F = 0, \quad (3)$$

$$\begin{aligned} A(x_1 + h)^2 + 2 B(x_1 + h)(y_1 + k) + C(y_1 + k)^2 \\ + 2 D(x_1 + h) + 2 E(y_1 + k) + F = 0. \end{aligned} \quad (4)$$

Removing parentheses in (4) and subtracting (3) member by member from the resulting equation, we have

$$\begin{aligned} 2 Ax_1h + Ah^2 + 2 By_1h + 2 Bx_1k + 2 Bhk \\ + 2 Cy_1k + Ck^2 + 2 Dh + 2 Ek = 0. \end{aligned}$$

On transposing and collecting we obtain an equation which may be put in the form

$$\begin{aligned} k(2 Bx_1 + 2 Cy_1 + Ck + 2 E) \\ = -h(2 Ax_1 + Ah + 2 By_1 + 2 Bk + 2 D). \end{aligned}$$

From this we have

$$\frac{k}{h} = - \frac{2 Ax_1 + Ah + 2 By_1 + 2 Bk + 2 D}{2 Bx_1 + 2 Cy_1 + Ck + 2 E}.$$

As k and h both approach zero the right member of this equation approaches

$$- \frac{Ax_1 + By_1 + D}{Bx_1 + Cy_1 + E}.$$

Hence k/h approaches this quantity. Therefore this quantity is equal to the slope m of the tangent to the given curve at $P_1(x_1, y_1)$. Substituting this value of m in equation (1) (Sec. 81) we have for the required tangent line the equation

$$y - y_1 = - \frac{Ax_1 + By_1 + D}{Bx_1 + Cy_1 + E} (x - x_1).$$

This equation may be reduced to simpler form. Clearing of fractions and putting the variable terms alone in the first member, we have

$$\begin{aligned} Ax_1x + Bx_1y + By_1x + Cy_1y + Dx + Ey \\ = Ax_1^2 + 2 Bx_1y_1 + Cy_1^2 + Dx_1 + Ey_1. \end{aligned} \quad (5)$$

From equation (3) it follows that the second member of this equation has the value

$$- Dx_1 - Ey_1 - F.$$

Replacing the second member of (5) by this value and simplifying, we have

$$\begin{aligned} Ax_1x + B(x_1y + xy_1) + Cy_1y + D(x + x_1) \\ + E(y + y_1) + F = 0. \end{aligned} \quad (6)$$

This is the equation of the tangent line at $P_1(x_1, y_1)$ to the curve defined by equation (2).

The result may be described as follows: The equation of the tangent at $P_1(x_1, y_1)$ to the curve whose equation is (2) may be obtained from that equation by replacing x^2 by x_1x , y^2 by y_1y , $2xy$ by $x_1y + xy_1$, $2x$ by $x + x_1$, and $2y$ by $y + y_1$.

83. Lengths of tangents, normals, subtangents, and subnormals. Tangents and normals to a curve are lines of indefinite extent. Certain segments of them are, however, spoken of as the **tangent** and the **normal**, and the lengths of these segments are called the **tangent length** and the **normal length** respectively. In the case of the tangent line this segment is the portion of the line included between the point of contact and the intersection of the line with the x -axis. In the case of the normal line the segment in consideration is that portion which is between the point on the curve and the intersection of the normal line with the x -axis. In the following figure PT is the tangent segment and PN is the normal segment corresponding to the point P on the curve. It is customary to take the lengths of PT and PN as positive.

If we draw the perpendicular PM to the x -axis, then the projection TM of TP on the x -axis is called the **subtangent**. The projection MN of PN on the x -axis is called the **subnormal**.

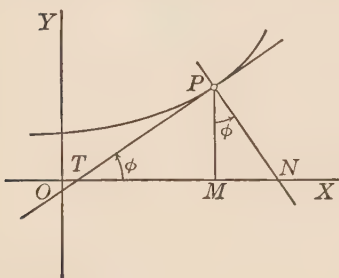


FIG. 76

Let ϕ denote the inclination angle of the tangent line. Then $\tan \phi$ is equal to the slope m of the tangent line TP . For a given point P on a given curve the slope m , and hence the value of $\tan \phi$, can be found by the method of Sec. 81. Moreover for a given point $P_1(x_1, y_1)$ the length of PM is known; it is y_1 .

Hence PT and TM can be found by trigonometry. Moreover the angle MPN is also equal to ϕ , since its sides are perpendicular to the sides of the angle MTP and both the angles are acute. Hence PM and MN can be found by trigonometry. Therefore we have a direct means for finding the lengths of the tangent, the normal, the subtangent, and the subnormal for a given curve at a given point.

EXAMPLE. Find the lengths of the tangent, the normal, the subtangent, and the subnormal for the parabola $y^2 = 2px$ at the point $P_1(x_1, y_1)$ on the parabola.

In Example 1 of Sec. 81 we saw that the slope of the tangent line is p/y_1 . Hence

$$\frac{p}{y_1} = \tan \phi = \frac{PM}{TM} = \frac{y_1}{TM}.$$

Therefore

$$\text{subtangent} = TM = \frac{y_1^2}{p} = 2x_1.$$

$$\text{Now } \overline{PT}^2 = \overline{TM}^2 + \overline{PM}^2.$$

$$\text{Hence } PT = \sqrt{4x_1^2 + y_1^2} = \sqrt{4x_1^2 + 2px_1}.$$

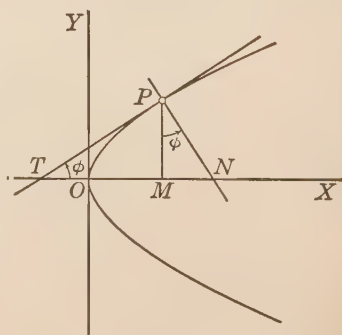


FIG. 77

Again, by aid of the triangle PMN , we have

$$\frac{p}{y_1} = \tan \phi = \frac{MN}{PM} = \frac{MN}{y_1}.$$

Hence

$$\text{subnormal} = MN = p.$$

Also, $\text{normal} = PN = \sqrt{PM^2 + MN^2} = \sqrt{y_1^2 + p^2}.$

84. The equation of a tangent line with a given slope. The methods for finding the equation of the tangent line if its slope is given will be illustrated by the solutions of some examples.

EXAMPLE 1. Find the equation of the tangent line to the parabola $y^2 = 2px$, the slope of the tangent being m .

First solution. In Example 1 of Sec. 81 we saw that the slope of the tangent at the point (x_1, y_1) is p/y_1 . Hence we must now have $p/y_1 = m$, or $y_1 = p/m$.

But $y_1^2 = 2px_1$. Hence $p^2/m^2 = 2px_1$; whence $x_1 = p/(2m^2)$. Therefore the point of tangency is $(p/2m^2, p/m)$. Since the slope of the tangent is m , we have for its equation

$$y - \frac{p}{m} = m\left(x - \frac{p}{2m^2}\right),$$

or
$$y = mx + \frac{p}{2m}.$$

Second solution. Since the slope of the tangent line is m , the equation of the line may be written in the form

$$y = mx + k,$$

where k is to be determined. In the figure we have drawn several lines with the same slope. It is evident that several of them cut the curve in two points, and that as the secant line with fixed slope m moves toward the tangent line the two points of intersection approach coincidence at the point of tangency. Hence we have to determine k so that the two

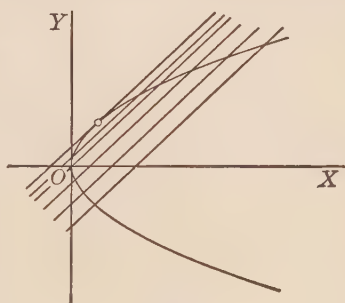


FIG. 78

equations $y^2 = 2px$ and $y = mx + k$ shall have two equal solutions. Eliminating y from the two equations, we have

$$(mx + k)^2 = 2px,$$

or
$$m^2x^2 + 2(km - p)x + k^2 = 0.$$

The roots of this equation are

$$\frac{p - km \pm \sqrt{(km - p)^2 - k^2m^2}}{m^2}.$$

In order that these two roots may be equal, the quantity under the radical must have the value zero. Hence we have

$$(km - p)^2 - k^2m^2 = 0.$$

Solving this equation for k , we have

$$k = \frac{p}{2m}.$$

Substituting this in the equation $y = mx + k$, we have

$$y = mx + \frac{p}{2m}$$

for the required equation of the tangent line.

EXAMPLE 2. Find the equation of the tangent line to the circle $x^2 - 10x + y^2 = 0$, the slope of the tangent being 1.

The equation of the tangent line has the form

$$y = x + k.$$

Substituting this value of y in the equation of the circle, we have

$$x^2 - 10x + (x + k)^2 = 0,$$

or
$$2x^2 + 2(k - 5)x + k^2 = 0.$$

The roots of this equation are

$$x = \frac{5 - k \pm \sqrt{(k - 5)^2 - 2k^2}}{2}.$$

In order that these roots shall be equal, the quantity under the radical must be equal to zero. Hence

$$(k - 5)^2 - 2k^2 = 0.$$

Solving this equation for k , we have

$$k = -5 + 5\sqrt{2} \quad \text{or} \quad k = -5 - 5\sqrt{2}.$$

Hence there are two tangent lines having the required property, and their equations are the following:

$$y = x - 5 + 5\sqrt{2}, \quad y = x - 5 - 5\sqrt{2}.$$

EXERCISES

1. Find the equations of the tangent and the normal to the circle $x^2 - 10x + y^2 = 0$ at the point $P_1(x_1, y_1)$ on the circle.

2. For what points (x_1, y_1) is the slope of the tangent line in Exercise 1 equal to 1? What are the equations of the tangent lines in these cases?

3. Find the equations of the tangent and normal lines and the lengths of the tangent, normal, subtangent, and subnormal for each of the following curves at the point indicated:

(a) $xy = 1$ at $(1, 1)$.

(b) $x^2 + y^2 = 10$ at $(3, 1)$.

(c) $x^2 - y^2 = 5$ at $(3, 2)$.

4. In the case of the following curves find the equations of the tangent lines with the given slopes:

(a) $xy = 12$, slope $= -3$.

(b) $y^2 = 8x + 12$, slope $= 1$.

(c) $x^2 - 10x + y^2 = 0$, slope $= -1$.

5. In the case of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

show that there are two tangents having the slope m and that their equations are

$$y = mx \pm \sqrt{a^2m^2 + b^2}.$$

6. In the case of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

show that there are two tangents having the slope m and that their equations are

$$y = mx \pm \sqrt{a^2m^2 - b^2}.$$

7. Find the equations of the tangent lines to the ellipse $4x^2 + 9y^2 = 36$ which are parallel to the line $x + 2y = 10$. Find those which are perpendicular to this line.

8. Show that the triangle formed by a tangent to a parabola, the directrix, and the line through the focus and the point of contact is an isosceles triangle.

9. By means of Exercise 8 show how to draw the tangent line to a parabola at a given point.

10. Let L be that chord of a parabola which is perpendicular to its axis at the focus. Show that the normal to the parabola at one end of L is parallel to the tangent at the other end.

11. Let FP be any line drawn from the focus F of a parabola to a point P on the parabola. Show that the circle on FP as a diameter touches the tangent line drawn to the parabola at its vertex.

12. Show that the normal to a parabola at any point bisects the angle formed by the line joining the point to the focus and the line through the point parallel to the axis of the parabola.

85. Diameters. A **diameter** of a conic is the locus of the middle points of any system of parallel chords. It will now be shown that this locus is a straight line.

Let us consider first the case of the parabola $y^2 = 2px$. Let m be the slope of each chord of a system of parallel chords. Then the equations of these chords are of the form $y = mx + k$, where k varies from chord to chord. Solving simultaneously the two equations

$$y^2 = 2px \quad \text{and} \quad y = mx + k,$$

we have for the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ of intersections of their graphs the coördinates

$$\begin{aligned} x_1 &= \frac{p - mk + \sqrt{p^2 - 2mpk}}{m^2}, & y_1 &= \frac{p + \sqrt{p^2 - 2mpk}}{m}, \\ x_2 &= \frac{p - mk - \sqrt{p^2 - 2mpk}}{m^2}, & y_2 &= \frac{p - \sqrt{p^2 - 2mpk}}{m}. \end{aligned}$$

Now the midpoint of the segment P_1P_2 is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Hence if we denote this midpoint by (x, y) we have

$$x = \frac{p - mk}{m^2}, \quad y = \frac{p}{m}.$$

Therefore y has the fixed value p/m , regardless of the value of k . Therefore *the locus of the middle points of the chords of slope m is the straight line*

$$y = \frac{p}{m}. \quad (7)$$

This is the equation of the diameter of the parabola which bisects the chords of slope m .

It is evident that every diameter of a parabola is parallel to the axis, and that every line parallel to the axis is a diameter.

In the case of the ellipse we proceed similarly. If the parallel chords have the slope m , then we have to solve simultaneously the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad y = mx + k.$$

For these solutions we have

$$\begin{aligned} x_1 &= \frac{-a^2mk + \sqrt{D}}{a^2m^2 + b^2}, & y_1 &= \frac{b^2k + m\sqrt{D}}{a^2m^2 + b^2}, \\ x_2 &= \frac{-a^2mk - \sqrt{D}}{a^2m^2 + b^2}, & y_2 &= \frac{b^2k - m\sqrt{D}}{a^2m^2 + b^2}, \end{aligned}$$

where D is an expression whose exact value we do not need. Using (x, y) to denote the point $((x_1 + x_2)/2, (y_1 + y_2)/2)$, we have

$$x = -\frac{a^2mk}{a^2m^2 + b^2}, \quad y = \frac{b^2k}{a^2m^2 + b^2}.$$

Hence
$$k = -\frac{(a^2m^2 + b^2)x}{a^2m}, \quad k = \frac{(a^2m^2 + b^2)y}{b^2}.$$

Equating these two values of k and simplifying the equation, we have

$$y = -\frac{b^2}{a^2m}x \quad (8)$$

as the equation of that diameter of the ellipse which bisects the system of chords of slope m . The minor axis is also a diameter.

One may treat in a similar way the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and thus show that the equation

$$y = \frac{b^2}{a^2 m} x \quad (9)$$

is the equation of that diameter of the hyperbola which bisects the system of chords of slope m .

In each case the equation of the diameter is linear in x and y , whence it follows that the diameter is a straight line. The diameter of an ellipse or a hyperbola passes through its center, as is evident from the form of the equation of the diameter. Moreover every chord through the center is a diameter, since m may evidently be chosen so that the equation of the diameter is identical with the equation $y = m_1 x$ of any line through the origin except the y -axis.

86. Conjugate diameters. If two diameters of an ellipse or a hyperbola are such that each of them bisects all the chords parallel to the other, the two diameters are said to be **conjugate**.

From equation (8) (Sec. 85) it follows that the diameter which bisects the chords of slope m in an ellipse has itself the slope $-b^2/(a^2 m)$. Hence if m and m' are the slopes of two conjugate diameters, we have

$$m' = -\frac{b^2}{a^2 m};$$

whence it follows that

$$mm' = -\frac{b^2}{a^2}.$$

In a similar manner it can be shown that if m and m' are the slopes of two conjugate diameters of a hyperbola, we have

$$mm' = \frac{b^2}{a^2}.$$

EXERCISES

1. Find the equation of that diameter of the ellipse $4x^2 + 25y^2 = 100$ which bisects the chords having the slope 2.

2. For the ellipse $3x^2 + 4y^2 = 12$ find the equation of the diameter which is conjugate to the diameter $2y + 3x = 0$.

3. A diameter of the parabola $y^2 = 8x$ passes through the point $(1, 1)$. Find its equation and also the equations of the chords which it bisects.

4. If (x_1, y_1) is one extremity of a diameter of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, prove that the extremities of the conjugate diameter are $(ay_1/b, -bx_1/a)$ and $(-ay_1/b, bx_1/a)$.

5. Let α and β be the lengths of two conjugate semidiameters of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. Show that $\alpha^2 + \beta^2 = a^2 + b^2$. (Use the result in the preceding exercise.)

6. Show that a tangent to an ellipse at an extremity of a diameter is parallel to the conjugate diameter.

7. Show that the four tangents to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at the extremities of any two conjugate diameters form a parallelogram whose area is $4ab$. (Use Exercises 4 and 6.)

8. Show that the tangent to a parabola at the point where a diameter cuts the parabola is parallel to the system of chords which are bisected by the diameter.

9. If the four lines are drawn each of which joins an extremity of the major axis to an extremity of the minor axis of an ellipse, show that they are parallel in pairs and that the two diameters parallel to them are conjugate diameters.

87. Poles and polars. If the point $P_1(x_1, y_1)$ is on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (10)$$

then we have seen (Sec. 81) that the tangent to the ellipse at the point P_1 has the equation

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1. \quad (11)$$

Now if $P_1(x_1, y_1)$ is taken to be any point (except the origin) in the plane of the ellipse, then equation (11) represents a straight line. This straight line is called the **polar** of the point

P_1 with respect to the ellipse, and the point P_1 is called the **pole** of the line with respect to the ellipse.

Similarly if $P_1(x_1, y_1)$ is any point (except the origin) in the plane of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad (12)$$

then the straight line

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1 \quad (13)$$

is called the polar of P_1 , and P_1 is said to be the pole of the line with respect to the hyperbola.

Likewise if $P_1(x_1, y_1)$ is any point in the plane of the parabola

$$y^2 = 2px, \quad (14)$$

then the line $y_1y = p(x + x_1)$ (15)

is called the polar of P_1 , and P_1 is said to be the pole of the line with respect to the parabola.

It is obvious that the polar of a point P_1 on a conic is the tangent to the conic at P_1 .

We shall now prove the following theorem :

If $P_2(x_2, y_2)$ is on the polar of $P_1(x_1, y_1)$ with respect to a conic, then P_1 is on the polar of P_2 with respect to the same conic.

If the conic is the ellipse (10), then the polar of P_1 has the equation

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1 \quad (16)$$

and the polar of P_2 has the equation

$$\frac{x_2x}{a^2} + \frac{y_2y}{b^2} = 1. \quad (17)$$

Since P_2 is on the locus of (16), we have

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = 1.$$

But this is the condition that P_1 is on the locus of (17). Hence the theorem is true in the case of the ellipse.

The student may prove in a similar way that the theorem is true for the hyperbola and the parabola.

If $P_1(x_1, y_1)$ is outside of a conic, then the polar of P_1 with respect to the conic passes through the points of contact of the tangents from P_1 to the conic.

Let us consider the case of the ellipse (10). If (x', y') is the point of contact of a tangent from P_1 to the ellipse, then (Sec. 81) the equation of the tangent is

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1.$$

Since P_1 is on this tangent, we have

$$\frac{x'x_1}{a^2} + \frac{y'y_1}{b^2} = 1.$$

But this is the condition that the point (x', y') is on the locus of (11), this locus being the polar of P_1 . Hence (x', y') is on the polar of P_1 . Therefore the polar of P_1 passes through the points of contact of the tangents from P_1 to the ellipse.

In a similar way the student may prove the theorem for the cases of the hyperbola and the parabola.

EXERCISES

1. Find the equation of the polar of $(2, 4)$ with respect to the ellipse $4x^2 + 9y^2 = 36$.

2. Find the equation of the polar of $P_1(x_1, y_1)$ with respect to the circle $x^2 + y^2 = r^2$.

3. Prove that the line which passes through the center of the circle $x^2 + y^2 = r^2$ and the point $P_1(x_1, y_1)$ is perpendicular to the polar of P_1 .

4. Show that the polar of a focus of a conic is a directrix of the conic.

5. Find the pole of the line $6x + y = 25$ with respect to the circle $x^2 + y^2 = 25$.

6. Find the pole of the line $ax + by + c = 0$ with respect to the parabola $y^2 = 2px$.

7. If a point P is inside the circle $x^2 + y^2 = r^2$, show that the polar of P lies wholly outside of the circle.

8. In the case of the circle $x^2 + y^2 = r^2$, whose center is O , let OP cut the polar of P in Q . Show that $OP \cdot OQ = r^2$.

9. If P is a point on a directrix of a conic, show that the polar of P passes through the corresponding focus.

10. Prove that the polar of P_1 with respect to an ellipse is parallel to that diameter of the ellipse which is conjugate to the diameter through P_1 .

CHAPTER VII

OTHER TYPES OF CURVES IN RECTANGULAR COÖRDINATES

88. Algebraic and transcendental curves. Equations (in rectangular coördinates) whose members can be expressed by a finite number of the operations of addition, subtraction, multiplication, and division of the variables are called **algebraic**. The equations $y^2 = x^3$, $x^2 + y^2 = 4$, and $y = (x-1)/(x+1)$, for example, are algebraic equations. Equations which cannot be so expressed are called **transcendental**. The equations $y = e^x$, $y = \log x$, and $x + \sin x = y^2$, for example, are transcendental equations. The graph of an algebraic equation in two variables is called an **algebraic curve**, and the graph of a transcendental equation in two variables is called a **transcendental curve**. Curves in a plane represented by algebraic equations of degree greater than two, or by transcendental equations, are called **higher plane curves**. The curves discussed in Chapters VII, VIII, and IX are higher plane curves, with a few exceptions, and are grouped into chapters according to the method of treatment.

ALGEBRAIC CURVES

89. Example. Discuss and graph $y^2 = (x-1)(x-2)(x-3)$.

The equation shows the locus to be symmetric with respect to the x -axis. The x -intercepts are 1, 2, and 3. For $x = 0$, $y^2 = -6$, and hence the curve does not intersect the y -axis. For $x < 1$, each factor in the right member is negative and there is no real value for y . For $1 < x < 2$, the first factor is positive and the other two are negative, y^2 is therefore positive and there are two real values of y . For $2 < x < 3$, y^2 is negative and there are no real values for y . For $x > 3$, there

are real values for y . As x increases indefinitely y increases without limit. Assigning a few values to x between 1 and 2 and values to x greater than 3 we find a few points on the curve, such as $(\frac{5}{4}, .5+)$, $(\frac{3}{2}, .6+)$, $(\frac{7}{4}, .5-)$, $(4, 2.4+)$, and $(5, 4.9-)$. The curve is shown in Fig. 79.

90. Example. Discuss and graph

$$y = \frac{x(x-1)}{(x+1)(x-3)}.$$

There is no symmetry. The curve passes through the origin and has an additional x -intercept, $x=1$. As x approaches -1 from the left y approaches $+\infty$, and as x approaches -1 from the right y approaches $-\infty$. As x approaches

3 from the left y approaches $-\infty$, and as x approaches 3 from the right y approaches $+\infty$. The lines $x=-1$ and $x=3$ are, therefore, vertical asymptotes of the curve. Dividing the numerator by the denominator, we have

$$y = 1 + \frac{x+3}{x^2-2x-3},$$

which shows that as x increases numerically y approaches 1, since $(x+3)/(x^2-2x-3)$ approaches zero.* For $x > 3$, $(x+3)/(x^2-2x-3)$ is positive, and therefore $y > 1$.

*A convenient method of finding the limit of a rational fraction as the variable becomes infinite is to divide numerator and denominator by the highest power of the variable which occurs and then evaluate the limit. For example,

$$\lim_{x \rightarrow \infty} \frac{x+3}{x^2-2x-3} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{3}{x^2}}{1 - \frac{2}{x} - \frac{3}{x^2}} = 0,$$

since each of the quantities $\frac{1}{x}$, $\frac{2}{x}$, $\frac{3}{x^2}$ approaches zero as x becomes infinite.

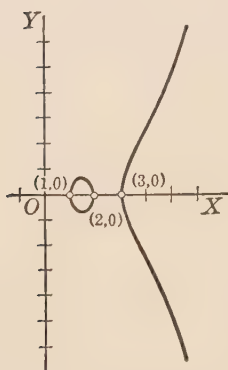


FIG. 79

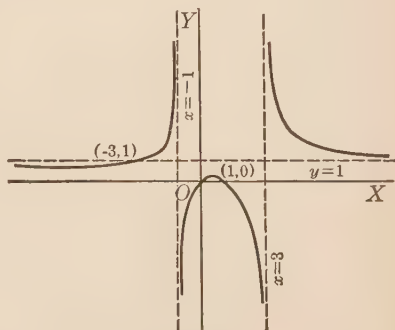


FIG. 80

For $x = -3, y = 1$. For $x < -3$, $(x+3)/(x^2-2x-3)$ is negative, and therefore $y < 1$. Hence the line $y = 1$ is an asymptote which the curve approaches from above toward the right and from below toward the left. The curve is composed of three branches. One branch is below the asymptote $y = 1$ at the left and remains between the x -axis and the line $y = 1$ until x reaches -3 , where it crosses the line $y = 1$ and rises indefinitely as it approaches the line $x = -1$. Another branch is on the right of the line $x = -1$ below the x -axis, goes through the origin, recrosses the x -axis at $x = 1$, and goes indefinitely downward as it approaches the asymptote $x = 3$. A third branch is on the right of the asymptote $x = 3$ and above the x -axis, comes downward and approaches the asymptote $y = 1$ without crossing it. Plotting the points with abscissas $-5, -4, -2, 2, 4, 5$, and 6 , drawing the asymptotes $y = 1, x = -1$, and $x = 3$, and indicating the intercepts on the axes, we have enough guides to draw the curve as shown in Fig. 80.

91. Graphing by composition of ordinates. It is sometimes possible to draw more easily the graph of a function by breaking it up into parts, graphing each part separately and combining the graphs. The method is illustrated by graphing the equation $y = x + 1/x$.

Let $y_1 = x$ and $y_2 = 1/x$, then $y = y_1 + y_2$. Graph $y_1 = x$ and $y_2 = 1/x$ on the same axes. Then add the ordinates corresponding to the same abscissas and the result will be the ordinate of $y = y_1 + y_2 = x + 1/x$.

In a similar manner the graph of a product may be found by graphing each factor and taking the product of the ordinates corresponding to the same abscissas. A quotient of two functions can be graphed in a similar manner.

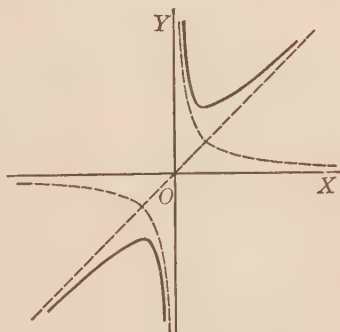


FIG. 81

EXERCISES

(The student should carefully reread Sec. 41 before attempting to discuss the following equations.)

Discuss and graph the following equations:

1. (a) $y = x(x - 3)^2$; (b) $y^2 = x(x - 3)^2$.

2. (a) $y = x^2(x - 2)$; (b) $y^2 = x^2(x - 2)$.

3. ~~(a)~~ $y = x(x - 1)(x - 2)(x - 3)$;
(b) $y^2 = x(x - 1)(x - 2)(x - 3)$.

4. (a) $y = x^3(x - 3)$; (b) $y^2 = x^3(x - 3)$.

5. (a) $y^2 = x^4 - 4x^2$; (b) $y^2 = 4x^2 - x^4$.

6. (a) $y^2 = \frac{4(x - 3)}{x}$; (b) $y^2 = \frac{4(3 - x)}{x}$.

7. $y = \frac{x(x + 2)}{x - 2}$.

10. $y^2 = \frac{x^2(2 + x)}{2 - x}$.

8. $y = \frac{2x(x + 1)}{x^2 - 4}$.

11. $y^2 = \frac{x^2 - 1}{(x - 3)^2}$.

9. ~~(a)~~ $y = \frac{x^2(x + 2)}{x - 2}$.

12. $y^3 = \frac{x^3}{x^3 - 1}$.

Sketch the graphs of the following equations by the method suggested in Sec. 91:

13. $y = x^2 + \frac{1}{x}$.

15. $y = x - \frac{1}{x}$.

17. $y = x \pm \sqrt{9 - x^2}$.

14. $y = \frac{x}{2} + \frac{1}{x^2}$.

16. $y = \frac{1}{x} + \frac{1}{x^2}$.

18. $y = x \pm \sqrt{x^2 - 4}$.

19. $y = 2x \pm 3\sqrt{x}$.

20. $25y^2 + 24xy + 16x^2 - 400 = 0$. (Solve for y in terms of x .)

TRANSCENDENTAL CURVES

92. The trigonometric functions $\sin x$, $\cos x$, and $\tan x$. From trigonometry we know that as x increases from 0 to $\pi/2$, $\sin x$ varies from 0 to 1; as x increases from $\pi/2$ to π , $\sin x$ varies from 1 to 0; as x increases from π to $3\pi/2$, $\sin x$ varies from 0 to -1 ; as x increases from $3\pi/2$ to 2π , $\sin x$ varies from -1 to 0; as x increases from 2π to 4π , $\sin x$ goes through the same cycle of values as when x in-

creases from 0 to 2π . We find exactly how $\sin x$ varies by consulting a table of sines. The curve $y = \sin x$ is represented in Fig. 82. The function $\cos x$ goes through the same cycle of

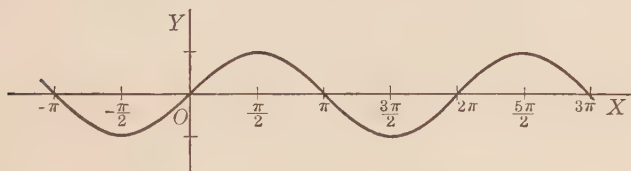


FIG. 82

values as $\sin x$, starting with the value 1 at $x = 0$. The curve $y = \cos x$ is shown in Fig. 83. The function $\tan x$ varies from

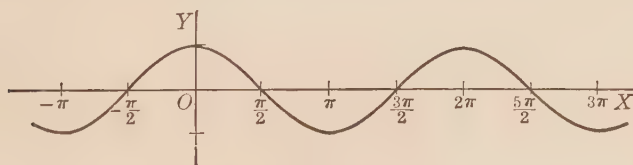


FIG. 83

$-\infty$ to $+\infty$ as x increases from $-\pi/2$ to $\pi/2$, and again as x increases from $\pi/2$ to $3\pi/2$, etc. (See Fig. 84.)

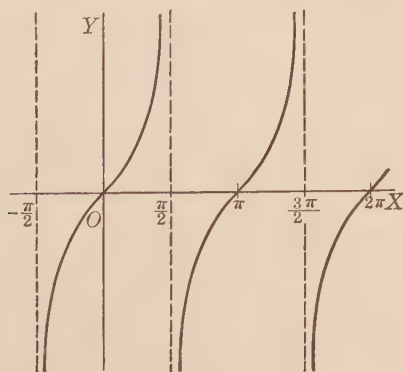


FIG. 84

93. Inverse trigonometric functions. Since the equations $y = \sin^{-1} x$, $y = \cos^{-1} x$, and $y = \tan^{-1} x$ are equivalent to the

equations $x = \sin y$, $x = \cos y$, and $x = \tan y$, respectively, the inverse trigonometric functions graph like the corresponding direct functions with the x - and y -axes interchanged. See Figs. 85, 86, and 87, respectively.

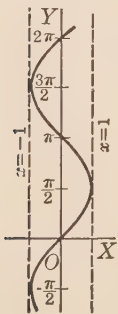


FIG. 85



FIG. 86

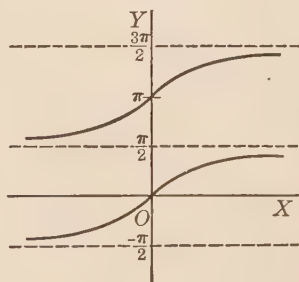


FIG. 87

EXERCISES

Sketch the graphs of the following equations:

1. (a) $y = 2 \sin x$; (b) $y = \sin 2x$; (c) $y = \sin \frac{x}{2}$.
2. (a) $y = \sin^2 x$; (b) $y^2 = \sin x$; (c) $y = \sin^3 x$; (d) $y^3 = \sin x$.
3. (a) $y = \cos 3x$; (b) $y = \cos \frac{3x}{2}$; (c) $y = \cos \frac{2x}{3}$.
4. (a) $y = \cos \left(x - \frac{\pi}{4} \right)$; (b) $y = \cos \left(x + \frac{\pi}{3} \right)$.
5. (a) $y = \tan 2x$; (b) $y = \tan^2 x$.
6. (a) $y = \sec x$; (b) $y = \csc x$; (c) $y = \cot x$.
7. (a) $y = \sin x + \cos x$; (b) $y = \sin x - \cos x$. (See Sec. 91.)
8. (a) $y = \sin 2x + \cos x$; (b) $y = \cos 2x + \sin x$.
9. (a) $y = \cos x - \frac{1}{2} \sin 2x$; (b) $y = \sin x - \frac{1}{2} \cos 2x$.
10. (a) $y = x + \sin x$; (b) $y = x + \sin 2x$.
11. (a) $y = \sin \pi x$; (b) $y = \sin \frac{\pi}{x}$.
12. (a) $y = \sec^{-1} x$; (b) $y = \csc^{-1} x$.
13. (a) $y = \sin^{-1} 2x$; (b) $y = \cos^{-1} \frac{x}{2}$.

94. The exponential function $y = a^x$ ($a > 0$). The curve crosses the y -axis at $y = 1$, but does not cross the x -axis. There are no values of x for which y is negative or zero, and hence the curve is entirely above the x -axis. For further discussion it is necessary to distinguish the cases where $a > 1$, $a = 1$, and $a < 1$.

For $a > 1$, as x increases through positive values y increases very rapidly. Hence the curve turns away from the x -axis much more rapidly, in the first quadrant, than it recedes from the y -axis. For x negative, y is less than 1. As x approaches $-\infty$, y approaches 0. The curve, therefore, approaches the x -axis as an asymptote as x approaches $-\infty$.

In case $a = 1$, $y = 1$ for every x and the curve is a straight line.

Since $a^x = (1/a)^{-x}$, it follows that for $0 < a < 1$ the curve is symmetric with respect to the y -axis to the curve $y = a_1^x$, where $a_1 = 1/a$. In Fig. 88, the curve $y = a^x$ is shown for $a = e = 2.718 \dots$, for $a = 10$, and for $a = 0.1$.

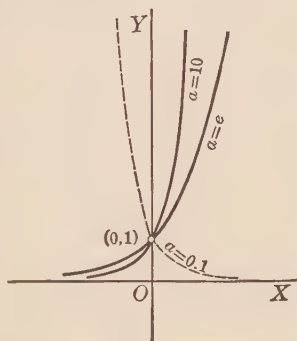


FIG. 88

95. Scientific applications of the exponential function. The exponential relation occurs frequently and in many fields.

(a) Bacterial growth proceeds according to an exponential law.

(b) Some chemical actions, such as the conversion of sugar under certain conditions, follow an exponential law.

(c) The decomposition of radium is according to an exponential law.

(d) The friction of a rope around a post, automobile braking, and a slipping belt involve an exponential law.

(e) The equation expressing Newton's law of cooling is exponential.

(f) The relation between atmospheric pressure and the height above the surface of the earth is exponential.

(g) The growth of current immediately after closing a circuit with constant electromotive force, resistance, and inductance is exponential.

(h) The amplitude of a damped vibration under certain restrictions decreases according to an exponential law.

(i) If we think of interest as being compounded continuously the amount is an exponential function of the time. For this reason the law expressed by an exponential relation

is often called the "compound interest law."

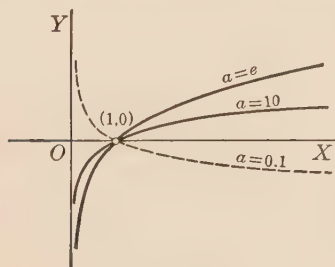


FIG. 89

96. The logarithmic function

$y = \log_a x$ ($a > 1$). Since $y = \log_a x$ is equivalent to $x = a^y$, the graph of the logarithmic function will be like that of the exponential function with the x - and y -axes interchanged.

Fig. 89 shows $y = \log_a x$ for $a = e = 2.718 \dots$ and for $a = 10$. The dotted curve in Fig. 89 represents $y = \log_{0.1} x$.

97. The graph of $y = e^{-x} \sin \pi x$. We draw the graphs of the equations $y_1 = e^{-x}$, $y_2 = \sin \pi x$, and $y_3 = -e^{-x}$ (Fig. 90). For integer values of x , $\sin \pi x = 0$, hence the curve $y = e^{-x} \sin \pi x$ crosses the x -axis at each point whose abscissa is an integer value of x . For $x = (4n + 1)/2$ (n an integer) $\sin \pi x = 1$, and the value of y coincides with the value of y_1 . And for $x = (4n + 3)/2$ (n an integer) $\sin \pi x = -1$, and the value of y coincides with the value of y_3 . For $0 < \sin \pi x < 1$, $y < y_1$, and for $-1 < \sin \pi x < 0$, $y > y_3$. Hence the curve $y = e^{-x} \sin \pi x$ lies between the

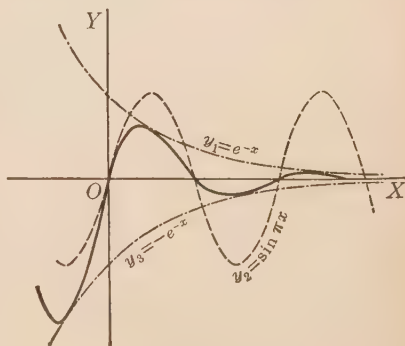


FIG. 90

curves $y_1 = e^{-x}$ and $y_3 = -e^{-x}$, touching the curve $y_1 = e^{-x}$ whenever $\sin \pi x = 1$ and touching the curve $y_3 = -e^{-x}$ whenever $\sin \pi x = -1$. The ordinate y for any abscissa is the product of the ordinates y_1 and y_2 for the same abscissa.

EXERCISES

Sketch the graphs of the following equations:

1. (a) $y = 2^x$; (b) $y = 2^{-x}$. 3. (a) $y = 3^x$; (b) $y = (\frac{1}{3})^x$.
2. (a) $y = e^x$; (b) $y = e^{x^2}$. 4. (a) $y = e^{2x}$; (b) $y = e^{\frac{x}{2}}$.
5. (a) $y = (0.1)^x$; (b) $y = 10^{\frac{1}{x}}$.
6. (a) $y = \log_5 x$; (b) $y = \log_5 2x$.
7. (a) $y = \log_e x$; (b) $y = \log_e x^2$.
8. (a) $y = \log_{10} (x+1)$; (b) $y = \log_{10} (x^2+1)$.
9. (a) $y = \log_{10} (x^2-1)$; (b) $y = \log_{10} (1-x^2)$.
10. (a) $y = x \cdot 2^x$; (b) $y = x^2 2^x$.
11. (a) $y = \frac{e^x + e^{-x}}{2}$; (b) $y = \frac{e^x - e^{-x}}{2}$.
12. $y = e^{-\frac{t}{10}} \cos \pi t$.

MISCELLANEOUS EXERCISES

Sketch the graphs of the following equations:

1. $y = \frac{8a^3}{x^2 + 4a^2}$ (the witch).
2. $y^2 = \frac{x^3}{2a-x}$ (the cissoid).
3. $y^2 = \frac{x^2(a-x)}{a+x}$ (the strophoid).
4. $y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right)$ (the catenary).
5. $y = e^{-x^2}$.
6. $\frac{x^2}{a^2} + \left(\frac{y}{b} \right)^{\frac{2}{3}} = 1$.
7. $y^2 = x(x-1)^2(x+2)(x-2)$.
8. $y = \frac{\sin x}{x}$.
9. $y = x \sin \frac{\pi}{x}$.
10. $y = (1+x)^{\frac{1}{x}}$.
11. $y = \log_{10} \frac{x-1}{x+1}$.

12. Show that the curves $y = \log_a nx$ ($n \neq 0$) have the same shape as the curve $y = \log_a x$. What is the relation of the curves $y = \log_a x^n$ to the curve $y = \log_a x$?

13. What is the general shape of the graph of $y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ (n a positive integer)?

14. In the equation of the catenary, Exercise 4, solve for x in terms of y .

15. Solve the equation $x = \log_e(y + \sqrt{y^2 \pm 1})$ for y in terms of x and graph. (See Exercise 11, Sec. 97.)

16. Find the equation of the locus of a point which moves so that the product of its distances from two fixed points is a constant. Call the distance between the fixed points $2c$ and the constant a^2 . (This curve is called the oval of Cassini.)

17. What does the equation of the locus in Exercise 16 become if $a = c$? (This curve is called the lemniscate of Bernoulli.)

18. Draw on the same axes the ovals of Cassini for $a = 5$, $c = 4$; $a = 4$, $c = 4$; $a = 3$, $c = 4$.

19. The triangle ABC has a fixed base $AB = 2c$. Find the locus of the vertex C in the following cases:

- (a) The sum of the sides AC and BC is a constant.
- (b) The difference of the sides AC and BC is a constant.
- (c) The product of the sides AC and BC is a constant.
- (d) The quotient of the sides AC and BC is a constant.

20. If P is a point on the line $5x - 2y + 4 = 0$ and the origin is at O , find the equation of the locus of Q , a point on OP such that $OP \cdot OQ = 16$. Draw the line and the locus of Q on the same axes. Show that the line and the locus of Q intersect on the circle $x^2 + y^2 = 16$.

21. If P is a point on the parabola $y^2 = 2px$ with vertex at O , find the equation of the locus of Q , a point on OP such that $OP \cdot OQ = p^2$.

22. If P is a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with center at O , find the equation of the locus of Q , a point on OP such that (a) $OP \cdot OQ = a^2$; (b) $OP \cdot OQ = b^2$. How are the new loci related to the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$?

23. If P is a point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with center at O , find the equation of the locus of Q , a point on OP such that $OP \cdot OQ = a^2$.

CHAPTER VIII

PARAMETRIC EQUATIONS

98. Parameters. It is sometimes convenient to express the rectangular coördinates of a point in terms of a third variable, or parameter.* Each rectangular coördinate is expressed in terms of the auxiliary variable and hence two equations are necessary to represent a locus. The elimination of the parameter between the two equations gives the equation in x and y . For example, the equations $y = t + 2$, $x = t^2 + 2t$ represent the curve shown in the accompanying figure. Points on the curve are found by assigning values to the parameter t and calculating the coördinates x and y from the given equations. Eliminating the parameter, we have $x = y^2 - 2y$. We see from this equation that the curve is a parabola.

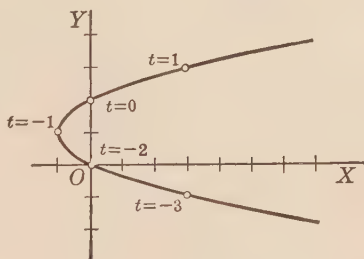


FIG. 91

An equation in x and y may be replaced by two parametric equations in an unlimited number of ways. It is sufficient to express one of the coördinates, x or y , in terms of a parameter in any way we choose, substitute in the equation and solve for the other coördinate in terms of the parameter. The equations for x and y thus obtained are parametric equations of the same locus as that defined by the equation in x and y . For example, if we put $y = t + 1$ in the equation $x = y^2 - 2y$, we obtain $x = t^2 - 1$. Hence $x = t^2 - 1$, $y = t + 1$ is another parametric representation of the same parabola graphed in Fig. 91.

* The word "parameter" is used to mean either an auxiliary variable or an arbitrary constant.

Parameters are frequently used in deriving equations of loci. Often the parameter is time or a geometric quantity, such as an angle. It should be observed that the locus is independent of the parameter used.

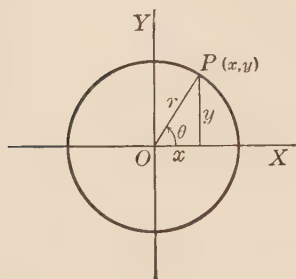


FIG. 92

99. Parametric equations of a circle. If we look upon the circle (Fig. 92) as traced by the point $P(x, y)$ moving continuously around the circumference, we may choose $\angle XOP = \theta$ as the parameter and write

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta. \quad (1)$$

These are parametric equations of a circle with center at the origin.

Squaring and adding the equations (1), we have

$$x^2 + y^2 = r^2(\sin^2 \theta + \cos^2 \theta) = r^2.$$

This is the rectangular equation of a circle with center at the origin and radius r . This equation and the parametric equations represent the same circle.

If we suppose that the point P moves at a constant rate, so that OP turns through an angle ω each second, then at the end of t seconds $\theta = \omega t$. Substituting this value for θ in equations (1), we have

$$x = r \cos \omega t \quad \text{and} \quad y = r \sin \omega t.$$

The parametric equations in the latter form enable us to locate the position of the point P in terms of the time during which it has been moving.

100. Parametric equations of an ellipse. Two circles with centers at the origin have radii a and b ($a > b$). A line is drawn from O cutting the outer circle at A and the inner circle at B . The line AN is drawn parallel to the y -axis and the line BD is drawn parallel to the x -axis. The lines AN and BD intersect at P . We shall show that the locus of P is an ellipse.

From the figure we have for the coördinates of P

$$x = ON = a \cos \phi \quad \text{and} \quad y = NP = MB = b \sin \phi.$$

Dividing by a and b , respectively, squaring and adding, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sin^2 \phi + \cos^2 \phi = 1.$$

This equation represents an ellipse with semiaxes a and b .

Hence

$$x = a \cos \phi,$$

$$y = b \sin \phi$$

are parametric equations of an ellipse.

101. The path of a projectile.

Let us assume that no forces

other than the attraction of the earth and the initial velocity affect the motion of the projectile. Choose the horizontal and vertical lines through the point at which the projectile leaves the propelling apparatus as the x - and y -axes, respectively.

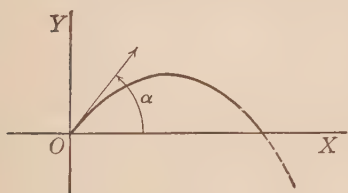


FIG. 94

Call the initial velocity v_0 , the angle the initial direction makes with the horizontal α , and the measure of the attraction of the earth g . If g is expressed in feet, then x , y , and v_0 must be expressed in feet.

The horizontal component of the initial velocity is $v_0 \cos \alpha$, and the vertical component is $v_0 \sin \alpha$. The horizontal distance traveled by the projectile in t seconds will be $v_0 t \cos \alpha$, and the vertical distance traveled will be $v_0 t \sin \alpha - \frac{1}{2} g t^2$. The vertical distance is made up of the distance traveled due to the initial velocity minus the distance the attraction of the earth pulls the projectile down. Hence the coördinates of P , after t seconds, will be

$$x = v_0 t \cos \alpha,$$

$$y = v_0 t \sin \alpha - \frac{1}{2} g t^2.$$

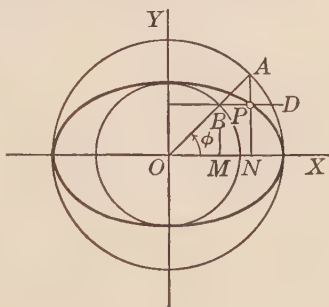


FIG. 93

These are parametric equations of the path of the projectile. That the path is a parabola is shown by the elimination of the parameter t , which gives

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2} \sec^2 \alpha.$$

EXERCISES

Draw the curves represented by the following parametric equations. Then eliminate the parameter and identify with curves previously studied if possible.

1. $x = 1 - t, y = 1 + t.$

6. $x = t^2, y = t^3.$

2. $x = 3t, y = t^2.$

7. $x = 4 \cos \theta, y = 4 \sin \theta.$

3. $x = 10/t^2, y = 10/t.$

8. $x = 4 \cos \theta, y = 5 \sin \theta.$

4. $x = t^2, y = t - 2.$

9. $x = a \sec \theta, y = b \tan \theta.$

5. $x = t, y = t^3.$

10. $x = 4(1 + \cos \theta), y = 3 \sin \theta.$

11. Show that $x = a + bt, y = a' + b't$ represent a straight line with slope b'/b .

12. Draw the line $x = 2 + 3t, y = 3 + 2t.$

13. Show that $x = a \cos t + b \sin t, y = a \sin t - b \cos t$ represent a circle. Find its center and radius.

14. Show that $x = a \cos nt + c, y = b \sin nt + d$ represent an ellipse. Find its center and semi-axes.

15. Find the parametric equations of the parabola $y^2 = 2px$ if the parameter is the slope of the line joining the origin to the point (x, y) on the parabola.

16. Find parametric equations of a point on the rim of a wheel 6 feet in diameter which turns on its axle at the rate of 30 revolutions per minute if the point starts from (a) the x -axis; (b) the y -axis; (c) the line $y = x$. (Use origin at center of wheel.)

17. What will the equations of the path of a projectile become if the projectile is fired horizontally?

18. A bullet is fired at an angle of elevation of 30° with an initial velocity of 1600 feet per second. Assuming that the bullet leaves the gun at the ground level and that $g = 32$, find how far away the bullet strikes the earth (assumed to be level). How far above the earth is the highest point of its path?

19. If the bullet in the preceding exercise strikes a target 784 feet above the earth, how far away in a horizontal direction is the target from the gun?

20. If a baseball leaves the hand of a pitcher $5\frac{1}{2}$ feet above the ground in a horizontal direction, what must be its initial velocity to cross the plate 18 inches above the ground? Is this a possible velocity?

21. The parametric equations of the path of a projectile are $x = 100t$, $y = 100t - 16t^2$. Plot the path of the projectile.

102. The cycloid.* *The cycloid is the path traced by a fixed point on the circumference of a circle as the circle rolls along a straight line.*

Let the line along which the circle rolls be the x -axis, and let one of the points at which the fixed point on the circle

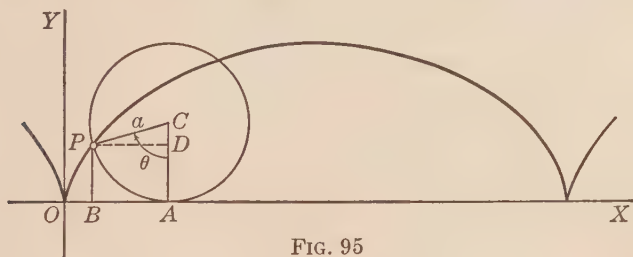


FIG. 95

comes in contact with the x -axis be the origin. Choose for the parameter the angle $\theta = \angle ACP$ (measured in radians) in Fig. 95. This is the angle through which the circle, of radius a , has rolled from its position when the point P was at O . The coördinates x and y of P are

$$x = OB = OA - BA = OA - PD,$$

$$y = BP = AD = AC - DC.$$

But $OA = \text{arc } AP = a\theta$, $PD = a \sin \theta$, $AC = a$, and $DC = a \cos \theta$. Substituting these values, we have

$$x = a\theta - a \sin \theta = a(\theta - \sin \theta),$$

$$y = a - a \cos \theta = a(1 - \cos \theta),$$

* Galileo (1564-1642) first called attention to the cycloid in 1630, suggesting that it be used for the arches of bridges.

as parametric equations of the cycloid. Eliminating θ between these equations, we have

$$x = a \cos^{-1} \frac{a-y}{a} - \sqrt{2ay - y^2}$$

as the equation in x and y . This is a transcendental equation.

103. Prolate and curtate cycloids. If the point P of the previous section is not on the circumference, the equation of the locus is obtained in a similar manner. Let b represent the

distance of the tracing point from the center of the rolling circle. Then the coördinates of the point P are

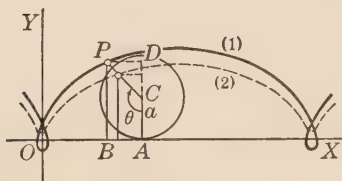


FIG. 96

$$x = OA - BA = a\theta - b \sin \theta,$$

$$y = AC - DC = a - b \cos \theta.$$

If $b > a$, the curve is called a **prolate** cycloid (Fig. 96, curve numbered (1)). If $b < a$, the curve is called a **curtate** cycloid (Fig. 96, curve numbered (2)).

104. Properties of the cycloid. The cycloid has many interesting properties. Some of these properties which can be obtained by more advanced mathematics are:

(a) The length of one arch is $8a$, that is, eight times the radius of the rolling circle.

(b) The area inclosed between one arch and the x -axis is $3\pi a^2$, that is, three times the area of the rolling circle.

(c) The teeth of gears are often cut with faces which are arcs of cycloids so that there is a rolling contact when the gears are in mesh.

If the circle rolls along the lower side of the fixed line the cycloid will be inverted, as in Fig. 97, and in this position has the additional properties:

(d) If two particles sliding without friction start from any two points of the curve, P_1 and P_2 , at the same time, they will reach the lowest point B at the same instant.

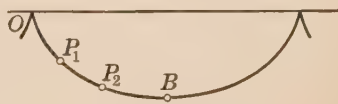


FIG. 97

(e) A particle sliding without friction will travel from O to B in less time than along any other curve connecting O and B . Hence the cycloid is sometimes called the curve of quickest descent.

The cycloid was the first transcendental curve to be rectified, that is, to have the length of its arc determined by mathematical means.*

EXERCISES

1. Sketch one arch of the cycloid $x = 10(\theta - \sin \theta)$, $y = 10(1 - \cos \theta)$ using values of θ at intervals of every 45° .
2. What are the lengths of the base and altitude of an arch of the cycloid in Exercise 1?
3. Derive the equations of the cycloid if the circle rolls along the under side of the fixed line.
4. Derive the equations of the cycloid if the origin is at the top of an arch.
5. Plot the curtate cycloid $x = 9\theta - 6\sin\theta$, $y = 9 - 6\cos\theta$.
6. Plot the prolate cycloid $x = 9\theta - 12\sin\theta$, $y = 9 - 12\cos\theta$.

105. The epicycloid. *If a circle rolls on the outside of another circle in the same plane, a fixed point on the circumference of the rolling circle will trace a path called an epicycloid.*

Let a be the radius of the fixed circle and b the radius of the rolling circle. The angle that the line of centers makes with the x -axis will be denoted by θ (Fig. 98) and the angle through which the radius CP turns from the line of centers will be denoted by ϕ ($\phi = \angle OCP$). Then $\angle ACO = \pi/2 - \theta$, and $\angle ACP = \phi - (\pi/2 - \theta) = \theta + \phi - \pi/2$. Arcs FE and PE are equal, hence $a\theta = b\phi$. The coördinates of the point P are $x = OA + AB = OA + DP = (a + b) \cos \theta + b \sin(\theta + \phi - \pi/2)$, $y = BP = AC - DC = (a + b) \sin \theta - b \cos(\theta + \phi - \pi/2)$.

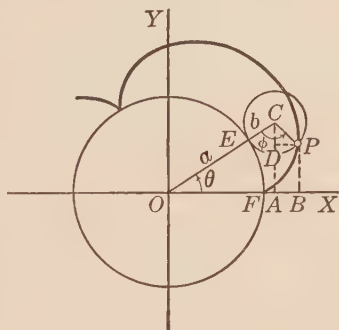


FIG. 98

* This was done by Sir Christopher Wren (1632-1723) in 1658.

But $\sin(\theta + \phi - \pi/2) = -\cos(\theta + \phi)$, $\cos(\theta + \phi - \pi/2) = \sin(\theta + \phi)$ and $\phi = a\theta/b$. Making these substitutions, we obtain the parametric equations of the epicycloid,

$$x = (a + b) \cos \theta - b \cos \frac{a+b}{b} \theta,$$

$$y = (a + b) \sin \theta - b \sin \frac{a+b}{b} \theta.$$

106. The hypocycloid. *If a circle rolls on the inside of a fixed circle in the same plane, a fixed point on the circumference of the rolling circle traces a curve called a hypocycloid.*

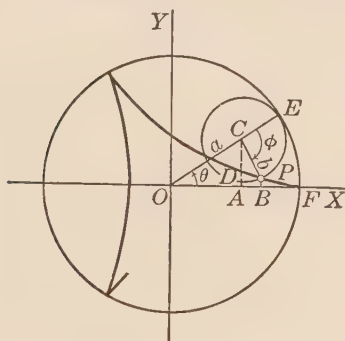


FIG. 99

The equations are obtained in a manner similar to that used for the epicycloid and differ from those equations only in the sign of b . Replacing b in the equations of the epicycloid by $-b$, we have the equations of the hypocycloid,

$$x = (a - b) \cos \theta + b \cos \frac{a-b}{b} \theta,$$

$$y = (a - b) \sin \theta - b \sin \frac{a-b}{b} \theta.$$

The special case where $a = 4b$ is of interest. The equations become

$$\begin{aligned} x &= 3b \cos \theta + b \cos 3\theta = b(3 \cos \theta + 4 \cos^3 \theta - 3 \cos \theta) \\ &= 4b \cos^3 \theta = a \cos^3 \theta, \end{aligned}$$

$$\begin{aligned} y &= 3b \sin \theta - b \sin 3\theta = b(3 \sin \theta - 3 \sin \theta + 4 \sin^3 \theta) \\ &= 4b \sin^3 \theta = a \sin^3 \theta. \end{aligned}$$

Eliminating θ , we get

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}(\cos^2 \theta + \sin^2 \theta),$$

or
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

where a is the radius of the fixed circle. This curve is called the hypocycloid of four cusps (Fig. 100).

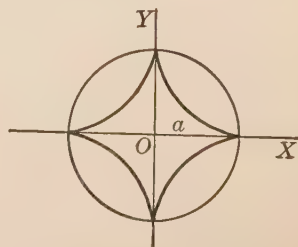


FIG. 100

EXERCISES

1. Sketch the epicycloid for

$$(a) \ b = \frac{a}{3}; \quad (b) \ b = \frac{a}{2}; \quad (c) \ b = \frac{2a}{3}; \quad (d) \ b = a.$$

2. Sketch the hypocycloid for

$$(a) \ b = \frac{a}{3}; \quad (b) \ b = \frac{a}{2}; \quad (c) \ b = \frac{2a}{3}; \quad (d) \ b = a.$$

3. Derive the parametric equations of the hypocycloid directly from a figure.

4. Show that the four-cusped hypocycloid is an algebraic curve.

107. The involute of a circle. *If a thread is unwound from around a fixed circle, a point on the thread traces a curve called the involute of the circle.*

Axes are chosen as indicated in Fig. 101, the point P on the thread having been in contact with the circle at A . The coördinates of P , expressed in terms of θ , give the parametric equations of the involute,

$$x = OD + DB = a \cos \theta + a\theta \sin \theta,$$

$$y = DC - EC = a \sin \theta - a\theta \cos \theta.$$

108. The folium of Descartes. The curve whose parametric equations are

$$x = \frac{3at}{1+t^3} \quad \text{and} \quad y = \frac{3at^2}{1+t^3}$$

is called the folium of Descartes.

Elimination of the parameter t gives the equation in the form

$$x^3 + y^3 = 3axy.$$

We shall discuss the curve from the parametric equations.

For positive values of t both x and y are positive and the curve lies in the first quadrant. For $t = 0$, $x = y = 0$, and for $t = 1$, $x = y = 3a/2$. For $0 < t < 1$, $x > y$, since for such values $t > t^2$, and for $t > 1$, $y > x$, since for such values

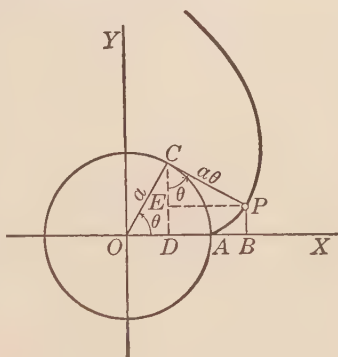


FIG. 101

$t^2 > t$. As t approaches ∞ , x and y approach zero.* Hence as t varies from 0 to ∞ , the curve makes a loop in the first quadrant, starting from and returning to the origin. For

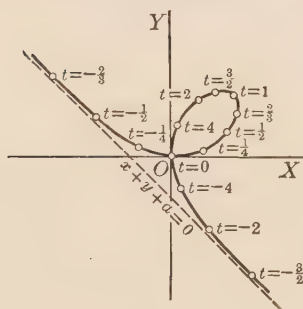


FIG. 102

for $-1 < t < 0$, $1 + t^3$ is positive, and therefore, for such values of t , x is negative and y positive. As t approaches -1 ($t > -1$), x approaches $-\infty$ and y approaches $+\infty$. Hence for values of t such that $-1 < t < 0$, the curve lies in the second quadrant. For values of $t < -1$, $1 + t^3$ is negative, and therefore, for such values of t , x is positive and y negative. As t approaches -1 ($t < -1$) x approaches $+\infty$ and y approaches $-\infty$. Hence for values of $t < -1$, the curve lies in the fourth quadrant. There are no values of t for which x and y are both negative, and hence no part of the curve lies in the third quadrant. As t passes through the value zero from negative values to positive values, y remains positive and x changes from negative to positive. Hence the curve goes through the origin without crossing the x -axis. As t approaches $+\infty$, y is positive and approaches zero and x is positive; and as t approaches $-\infty$, y is negative and approaches zero and x is positive. Hence the curve passes through the origin without crossing the y -axis.

As t approaches -1 , x and y approach either $+\infty$ or $-\infty$, but the sum,

$$x + y = \frac{3at}{1+t^3} + \frac{3at^2}{1+t^3} = \frac{3at(1+t)}{1+t^3} = \frac{3at}{1-t+t^2},$$

approaches $-a$. Hence as t approaches -1 , the curve approaches the line

$$x + y = -a$$

as an asymptote. A few points obtained by giving values to t , with the facts discussed above, enable us to draw the curve as shown in Fig. 102.

* See footnote, page 136.

MISCELLANEOUS EXERCISES

Eliminate the parameter from the following equations and identify the curves if possible:

1. $x = \cos 2\theta, y = \cos \theta.$

2. $x = \cos \frac{\theta}{2}, y = 1 + \cos \theta.$

3. $x = 5 \cos \theta + 3, y = 5 \sin \theta - 4.$

4. $x = e^t, y = 2e^{-t}.$

5. $x = t + \frac{1}{t}, y = t - \frac{1}{t}.$

6. $x = \sin \theta, y = \sin 2\theta.$

7. $x = \tan \theta, y = \sin \theta.$

8. $x = 2t^2 - 5t + 2, y = t - 2.$

9. $x = \cos \theta + \sin \theta, y = \cos \theta - \sin \theta.$

10. $x = t \cos t, y = t \sin t.$

11. $x = e^t \cos t, y = e^t \sin t.$

12. $x = \frac{t}{1+t^2}, y = \frac{t}{1-t^2}.$

13. $x = 2 \cos \theta - \cos 2\theta, y = 2 \sin \theta - \sin 2\theta.$

14. $x = a(1 + \cos \theta) \sin \theta, y = a \sin^2 \theta.$

15. $x = a \sin \theta, y = b \cos^3 \theta.$

16. $x = 2 \sin 3t, y = 3 \cos 2t.$

17. Show that $x = \frac{a(1-t^2)}{1+t^2}, y = \frac{2bt}{1+t^2}$ represent an ellipse.

18. Show that $x = a^t + a^{-t}, y = a^t - a^{-t}$ represent an equilateral hyperbola.

19. Show that $x = t^2 + t, y = t^2 - t$ represent a parabola.

20. Show that $x = a \log_e t, y = \frac{a}{2} \left(t + \frac{1}{t} \right)$ represent the catenary $y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right).$

21. Show that each of the following sets of parametric equations, (1) $x = a \cos^4 t, y = a \sin^4 t$, (2) $x = a \sec^4 t, y = a \tan^4 t$, and (3) $x = a \tan^4 t, y = a \sec^4 t$, represents a different arc of the same parabola.

22. Eliminate θ from the parametric equations of the cycloid.
23. If a ring with polished inner surface is laid on a table and the light shines from one side and above, a brightly illuminated curve is seen on the table within the ring. A similar curve may be observed in a glass of milk. Such a curve is called a caustic. If the ring has a radius of 1, the equations of the caustic are $x = \frac{3}{4} \cos \theta - \frac{1}{4} \cos 3\theta$, $y = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$. Draw the curve. Show that this curve is an epicycloid for which $a = \frac{1}{2}$ and $b = \frac{1}{4}$.
24. The rod OAB is pivoted at O and hinged at A . The part of the rod OA turns about O at a constant rate, while AB turns about A at a rate twice as great. Find the equations of B if $OA = AB$ and the turning begins when OAB is a straight line.
25. Two perpendicular lines MN and RS intersect at O . From a fixed point A on MN a line is drawn cutting RS at B . In either direction from B along AB the distance $BP = BO$ is laid off. The locus of P is called a strophoid. Find its equations.
26. A circle with diameter a is drawn tangent to the x -axis at the origin and cuts the y -axis again at B . Through B the line MN is drawn parallel to the x -axis. A line through the origin cuts the circle at C and MN at D . Lines through C and D parallel to the x - and y -axes, respectively, intersect at P . The locus of P is called the witch of Agnesi. Find the equations of the locus of P .
27. An insect crawls out along the spoke of a wheel 4 feet in diameter at the rate of 1 foot per minute. The wheel is rolling along the ground at the rate of 2 revolutions per minute. Find the equations of the path of the insect. Draw the graph showing the path of the insect for the first 2 minutes.
28. A crank OP , 2 feet in length, turns around O at a uniform rate. A rod 6 feet long is attached at P and the other end Q moves along a fixed line OB . Find the equations of the path of P , and the equations of the path of Q , if the origin is at O and the line OB is the x -axis.
29. The parallelogram $OQPQ'$ is hinged and OQ and OQ' turn about O at constant rates. Find the equations of the locus of P .

CHAPTER IX

POLAR EQUATIONS

109. Remarks. The equations of the straight line and the conics have been derived in polar coördinates. A few other curves were plotted from their polar equations in Chapter III. It is the purpose of this chapter further to illustrate the use of polar coördinates in the study of geometry. We have developed two coördinate systems. The student should observe that for some types of curves the equations are simpler in rectangular coördinates and for other types of curves the equations are simpler in polar coördinates. We are interested in coördinate systems only as a means. Hence that system of coördinates should be chosen which gives the greater algebraic simplicity. Several types of problems to which polar coördinates are particularly adapted are discussed in this chapter in the hope that the student may acquire ability in the choice of the proper coördinate system.

110. The rose-leaved curves

$\rho = a \sin n\theta$ and $\rho = a \cos n\theta$.

These families of curves are very much alike. We shall discuss the curve $\rho = a \sin 3\theta$.

The radius vector ρ will be zero whenever 3θ is 0° , 180° , or some multiple of 180° . This will occur when θ is 0° , 60° , 120° , 180° , 240° , 300° , etc. Numerically

ρ will be greatest when 3θ is some odd multiple of 90° , that is, when θ is 30° , 90° , 150° , 210° , 270° , 330° , etc. As θ varies from 0° to 30° , 3θ varies from 0° to 90° , $\sin 3\theta$ varies

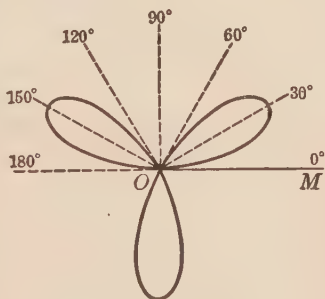


FIG. 103

from 0 to 1, and ρ varies from 0 to a . As θ varies from 30° to 60° , ρ decreases from a to 0; as θ varies from 60° to 90° , ρ decreases from 0 to $-a$; as θ varies from 90° to 120° , ρ

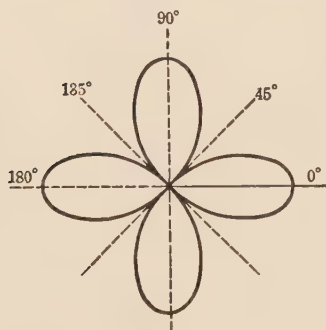


FIG. 104

increases from $-a$ to 0; as θ varies from 120° to 150° , ρ increases from 0 to a ; and as θ varies from 150° to 180° , ρ decreases from a to 0. As θ varies from 180° to 360° , the same curve is retraced. The graph of the equation is shown in Fig. 103.

In case n is an even number, the curve will have $2n$ loops, as is illustrated in Fig. 104 for $\rho = a \cos 2\theta$.

111. The limaçon of Pascal.* If to each radius vector of the circle, located with respect to the pole and the polar axis as shown in Fig. 105, is added a positive constant b , the locus of P , the end of the new radius vector, is a limaçon.

The equation of a circle thus located is (Sec. 56)

$$\rho = 2a \cos \theta,$$

where a is the radius. The equation of the locus of P is evidently

$$\rho = 2a \cos \theta + b.$$

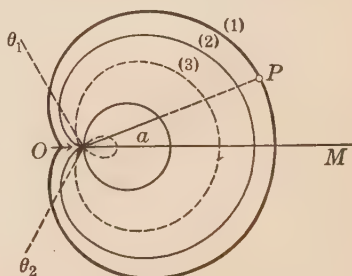


FIG. 105

(1) If $b > 2a$, ρ is always positive and the graph of the equation is shown in Fig. 105 as the curve numbered (1).

(2) If $b = 2a$, $\rho = 0$ for $\theta = \pi$ and is positive for other values of θ , and the resulting curve is shown as the curve numbered (2). This form of the limaçon is called the **cardioid**.

* Pascal was a French mathematician who lived from 1623 to 1662.

(3) If $b < 2a$, $\rho = 0$ for $\theta_1 = \cos^{-1}(-b/2a)$ in the second quadrant and for $\theta_2 = \cos^{-1}(-b/2a)$ in the third quadrant, and $\rho < 0$ for values of θ such that $\theta_1 < \theta < \theta_2$. The curve is shown as the curve numbered (3). For $b = a$, the curve is called the *trisectrix*.

112. The spirals. The spirals are curves for which the radius vector continuously increases or decreases as the vectorial angle changes. The curve makes a circuit of the pole for each change of 2π in the vectorial angle. The spirals are transcendental curves.

Some of the spirals have names suggested by the fact that the polar coördinates enter into the equation in the same way as the rectangular coördinates enter into the equation of the curve for which the spiral is named. For example, the equation $\rho^2 = a\theta$ is called the *parabolic spiral*. There is no analogy between the loci.

The spirals can be readily graphed by locating a few points on each turn of the curve about the pole. Fig. 106

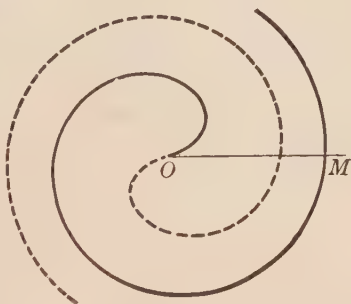


FIG. 106

shows the graph of $\rho^2 = 4\theta$. Solving for ρ , we have $\rho = \pm 2\sqrt{\theta}$. The solid line represents the part of the curve obtained by using positive values of ρ and the dotted line represents that part of the curve obtained by using negative values of ρ .

EXERCISES

Graph the rose-leaved curves corresponding to the following equations:

1. $\rho = a \sin 2\theta$.

2. $\rho = 4 \cos 3\theta$.

3. $\rho = a \sin 4\theta$.

4. $\rho = 4a \sin \theta$.

5. $\rho = a \cos 4\theta$.

6. $\rho = a \cos 5\theta$.

7. $\rho = 2 \sin 6\theta$.

8. $\rho = a \cos 10\theta$.

Graph the limaçons corresponding to the following equations:

9. $\rho = 1 - 2 \sin \theta$.

11. $\rho = 1 - 2 \cos \theta$.

10. $\rho = 2 - \sin \theta$.

12. $\rho = 2 - \cos \theta$.

Graph the spirals corresponding to the following equations:

13. $\rho = a\theta$ (spiral of Archimedes).

14. $\rho = e^{a\theta}$ (logarithmic).

16. $\rho^2\theta = a$ (lituus).

15. $\rho\theta = a$ (hyperbolic).

17. $(\rho - a)^2 = a^2(1 - \theta^2)$.

Transform to equations in rectangular coordinates:

18. $\rho\theta = a$.

19. $\rho = a \cos 2\theta$.

20. $\rho = a(1 - \sin \theta)$.

21. Find the polar equation of the locus of a point P which moves so that the radius vector OP varies directly as the vectorial angle.

113. The intersection of polar curves. Care must be exercised in the simultaneous solution of polar equations. It is usually desirable to check the solutions found by solving the equations simultaneously with the solutions found by drawing the two loci on the same diagram. The following example illustrates the difficulty.

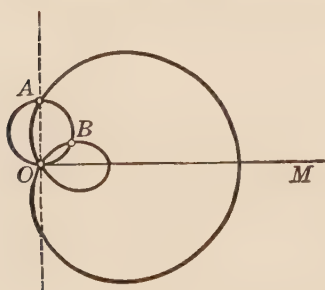


FIG. 107

EXAMPLE. Find the intersections of the curves $\rho = \sin \theta$ and $\rho = 2 \cos \theta + 1$.

Eliminating ρ , we have

$$\sin \theta = 2 \cos \theta + 1.$$

Replacing $\sin \theta$ by $\pm \sqrt{1 - \cos^2 \theta}$, squaring, and solving, we obtain

$$\cos \theta = 0 \quad \text{and} \quad \cos \theta = -\frac{4}{5}.$$

For $\cos \theta = 0$, we have from each equation $\rho = 1$ when $\theta = \pi/2$, and $\rho = -1$ and $\rho = 1$, respectively, when $\theta = 3\pi/2$. Hence $A(1, \pi/2)$ is an intersection. For $\cos \theta = -\frac{4}{5}$, θ in the second quadrant, $\rho = \frac{3}{5}$ in one case and $\rho = -\frac{3}{5}$ in the other. Hence there is no intersection for this value of θ . For $\cos \theta = -\frac{4}{5}$, θ in the third quadrant, $\rho = -\frac{3}{5}$ in each equation, and the point $B(-\frac{3}{5}, 216^\circ 52')$ is an inter-

section. The curves evidently intersect at O , but the simultaneous solution of the equations does not indicate the fact.

In some cases there are two or more points of intersection whose coördinates cannot be found by a simultaneous solution of the equations.

EXERCISES

Find the points of intersection of the following curves and check graphically :

1. $\rho = 2a \cos 2\theta$ and $\rho = a$.
2. $\rho = a\theta$ and $\rho\theta = a$.
3. $\rho^2 = \tan \theta \sec^2 \theta$ and $\rho = \sqrt{2}$.
4. $(\rho - a)^2 = a^2(1 - \theta^2)$ and $\rho = a\theta$.
5. $\rho = 6 \sin 2\theta$ and $\rho = 6 \cos 2\theta$.
6. $\rho = a \sin 6\theta$ and $\rho = a \cos 3\theta$.
7. $\rho = a(1 + 2 \cos \theta)$ and $\rho = a \cos \theta$.
8. $\rho = 4 \cos \theta$ and $\rho = 4 \sin \theta$.
9. $\rho^2 = a^2 \sin \theta$ and $\rho = a$.
10. $\rho = a(1 - \cos \theta)$ and $\rho = a \cos \theta$.
11. $\rho = a \cos \theta$ and $\rho = a \sin 2\theta$.
12. $\rho = 4(1 + 2 \cos \theta)$ and $\rho = 4 \sin \theta$.
13. $\rho = a(1 + \cos \theta)$ and $\rho = a \cos \frac{\theta}{2}$.
14. $\rho = a \sin^2 \frac{\theta}{3}$ and $\rho = a \sin^3 \frac{\theta}{3}$.

114. The cissoid of Diocles.* To a circle of radius a , with a diameter AB , a tangent is drawn at B . Any line is drawn from A cutting the circle at C and the tangent at D . The locus of P , where $AP = CD$, is called the cissoid.

Let A be the pole (Fig. 108), let AB be the polar axis, and let $\angle BAD = \theta$ be the vectorial angle. The $\angle BCD$ is a right angle and $\angle CBD = \theta$. From the figure we have

$$\rho = AP = CD = BD \sin \theta,$$

or

$$\rho = 2a \tan \theta \sin \theta,$$

as the polar equation of the cissoid.

* Diocles was a Greek mathematician who lived about 180 B.C.

The cissoid is an algebraic curve, as is shown by its equation in rectangular coördinates. Making the transformation to rectangular coördinates, we have

$$\sqrt{x^2 + y^2} = 2a \frac{y}{x} \frac{y}{\sqrt{x^2 + y^2}}.$$

On clearing of fractions and solving for y^2 , we have

$$y^2 = \frac{x^3}{2a - x}.$$

The equation in this form shows that the cissoid is symmetric with respect to the x -axis, that x cannot be less than zero nor greater than $2a$, and that $x = 2a$ is an asymptote to the curve.

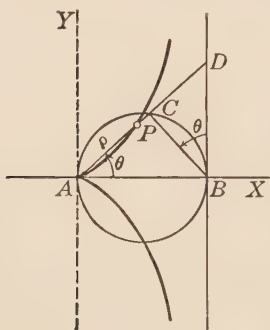


FIG. 108

115. The conchoid of Nicomedes.* The conchoid may be defined as follows: From a fixed point A , any line AB is drawn cutting a fixed line CD at M . The points P and P' are located so that $P'M = MP = b$, a constant. The locus of

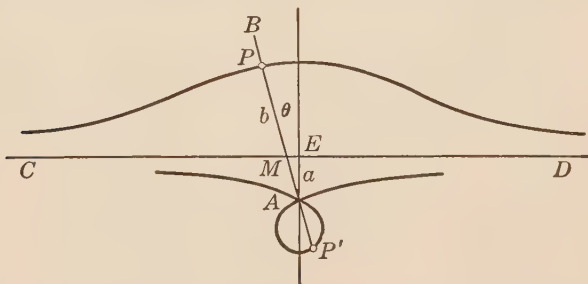


FIG. 109

P and P' is the curve called the conchoid. The fixed point A is called the pole and the fixed line CD is called the directrix of the curve.

Let A be the pole and AE , drawn perpendicular to CD , be the polar axis. Since the line CD and the point A are fixed, the distance AE is a constant which we call a . From the figure,

*Nicomedes was a Greek mathematician who lived at about the same time as Diocles.

$$AP = AM + MP = a \sec \theta + b,$$

$$AP' = AM - P'M = a \sec \theta - b.$$

The polar equation of the conchoid is, therefore,

$$\rho = a \sec \theta \pm b.$$

Since $\sec(-\theta) = \sec \theta$, the curve is symmetric with respect to the polar axis. As θ approaches $\pi/2$, or $-\pi/2$, $\sec \theta$ approaches ∞ and each of the two branches of the curve approaches the fixed line CD as an asymptote.

Transforming from polar to rectangular coördinates, we have

$$\sqrt{x^2 + y^2} = \frac{a\sqrt{x^2 + y^2}}{x} \pm b.$$

Clearing of fractions, rearranging, and squaring, we obtain

$$(x - a)^2(x^2 + y^2) = b^2x^2.$$

Hence the conchoid is an algebraic curve.

116. Problem. Find the locus of the vertex of a triangle with a fixed base if the product of the other two sides is equal to the square of one half the base.

Let AB (Fig. 110) be the fixed base, let the midpoint O of AB be the pole, and let OBM be the polar axis. We may designate the coördinates of B and A by $(a, 0)$ and (a, π) , respectively, where $2a$ is the length of the given base of the triangle. By the conditions of the problem

$$BP \cdot AP = a^2.$$

But $BP = \sqrt{\rho^2 + a^2 - 2a\rho \cos \theta}$
and $AP = \sqrt{\rho^2 + a^2 + 2a\rho \cos \theta}$;
hence we have

$$\sqrt{\rho^2 + a^2 - 2a\rho \cos \theta} \cdot \sqrt{\rho^2 + a^2 + 2a\rho \cos \theta} = a^2.$$

Squaring and reducing, we obtain

$$\begin{aligned} \rho^2 + 2a^2 - 4a^2 \cos^2 \theta &= \rho^2 + 2a^2(1 - 2 \cos^2 \theta) \\ &= \rho^2 - 2a^2 \cos 2\theta = 0, \end{aligned}$$

or

$$\rho^2 = 2a^2 \cos 2\theta,$$

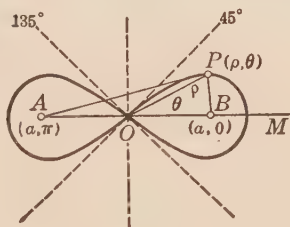


FIG. 110

as the equation of the required locus. Graphing this equation we obtain the required locus as shown by the curve in the figure. This curve is called the lemniscate of Bernoulli. It is a special case of the oval of Cassini ($a = c$ in Exercise 17, page 144).

MISCELLANEOUS EXERCISES

Graph the curves corresponding to the following equations:

1. (a) $\rho \sin \theta = 2$. (b) $\rho = 4 \sec \theta$. (c) $\rho \cos\left(\theta - \frac{\pi}{4}\right) = 3$.
2. (a) $\rho = 7$. (b) $\rho = 6 \cos \theta$. (c) $\rho = -4 \sin \theta$.
3. (a) $\rho = \frac{2}{\cos \theta + 1}$. (b) $\rho = \frac{3}{2 + \cos \theta}$. (c) $\rho = \frac{8}{1 + 2 \cos \theta}$.
4. $\rho = 2 - \sin 2\theta$.
5. $\rho = a^2 \tan^2 \theta \sec \theta$.
6. (a) $\rho^2 = a^2 \sin \theta$. (b) $\rho^2 = a^2 \sin 2\theta$. (c) $\rho^2 = a^2 \sin 4\theta$.
7. (a) $\rho = 4 \cos \frac{\theta}{2}$. (b) $\rho = 4 \sin \frac{2\theta}{3}$. (c) $\rho = 4 \cos \frac{4\theta}{3}$.
(d) $\rho = 4 \cos \frac{3\theta}{4}$.
8. (a) $\rho = a \sin^2 \frac{\theta}{2}$. (b) $\rho = a \cos^3 \frac{\theta}{3}$.

Transform into equations in rectangular coördinates:

$$9. \rho = \frac{a}{\sin \theta} + \frac{a}{\cos \theta}. \quad 10. \rho = a^2 \cot^2 \theta \csc \theta.$$

11. Show that $\rho^2 \sin 2\theta = a$ and $\rho^2 \cos 2\theta = a$ each represents an equilateral hyperbola.

Transform into equations in polar coördinates:

$$12. y = \frac{a^3}{x^2 + a^2}. \quad 13. (x^2 + y^2)^3 = 16 a^2 x^2 y^2 (x^2 - y^2)^2.$$

14. From the parametric equations of the strophoid obtained in Exercise 25, page 156, obtain its equation in polar coördinates.

15. A moving line segment of variable length has its ends on two fixed perpendicular lines and is at a constant distance from their point of intersection. Find the equation of the locus of its midpoint.

16. A moving line of constant length has its ends on the x - and y -axes. Find the polar equation of the locus of its midpoint.

17. A moving line of constant length has its ends on the x - and y -axes. Find the polar equation of the locus of any fixed point on it.

18. A moving line of constant length has its ends on the x - and y -axes. Find the polar equation of the locus of the foot of the perpendicular from the origin to the line.

19. Perpendiculars are dropped from a fixed point on a circle to the tangents to the circle. Find the locus of the feet of these perpendiculars.

20. If a line is drawn from a fixed point O outside a circle, cutting the circle in the points P and Q , show that $OP \cdot OQ$ is a constant.

21. In a triangle the base AB is a fixed length a . Find the equation of the locus of the vertex C if the exterior angle at B formed by producing AB is $\frac{3}{2} \angle CAB$. (Choose A for the pole and use the law of sines.)

22. In the equations of the epicycloid for $a = b$, translate the origin to $(a, 0)$, then change to polar coördinates and show that this form of the epicycloid is a cardioid.

23. Find the equation of the lemniscate of Bernoulli by changing the result of Exercise 17, page 144, to polar coördinates.

24. Show that all chords drawn through the pole of a cardioid are of equal length.

CHAPTER X

EMPIRICAL EQUATIONS

117. Origin of the problem. Many scientific laws may be obtained by a logical process of analysis from general principles. There are others which are obtained by a direct *empirical* method. That is to say, measurements, or observations, are made of the two or more related variables and the results are tabulated; then the table is subjected to a systematic examination. In the case where there are two related variables x and y , for instance, one tabulates the corresponding values of x and y obtained experimentally. Then it is desired to find a mathematical relation between x and y which is in agreement with the values in the table. This mathematical relation will express the law of connection between the related variables.

After the table has been constructed the process of finding the law is a purely mathematical one. It may be rendered easier by certain physical considerations which are available; but all its essential features are mathematical in character. No general method of solving this mathematical problem can be given here; but there are certain special and important cases of it which may readily be treated.

118. The linear law. The simplest case arises when the two variables x and y are so related that the law connecting them is expressed by means of a linear equation. If the given values of x and y are taken as rectangular Cartesian coördinates the corresponding points may be plotted. If these lie on a straight line the relation connecting x and y is linear, and the actual equation may be found by means of any two pairs of corresponding values of x and y . In practice, owing to errors in measurement, the points will in general not lie

precisely on a straight line. If they all lie very near to some straight line the linear law is assumed to hold and one finds the equation of a line which passes very near to all the points in consideration. The equation so obtained is taken as the approximate expression of the law involved.

We shall illustrate the method of dealing with this problem by solving some examples.

EXAMPLE 1. Let the corresponding values of x and y be those given by the following table:

$y =$	2.55	2.80	3.05	3.30	3.55	3.80	4.05
$x =$	.5	1	1.5	2	2.5	3	3.5

Find the equation connecting x and y .

The student should plot on ordinary coördinate paper the points corresponding to the pairs (x, y) as given in the table. He will then find that they lie (at least approximately) on a straight line. In order to determine that line take two of the points, say the points $(1, 2.8)$ and $(3, 3.8)$, and find the equation of the line through them. We have (see Sec. 17)

$$y - 2.8 = \frac{3.8 - 2.8}{3 - 1} (x - 1).$$

On simplifying, we have

$$y = .5x + 2.3.$$

The student should verify that each of the pairs (x, y) in the table satisfies this equation.

EXAMPLE 2. A wire suspended at one end was given successive loads W (measured in pounds) attached at the other end and the elongations E in inches of the wire were tabulated as follows:

$W =$	1	2	4	6	9	10	14
$E =$	.13	.26	.51	.77	1.15	1.28	1.79

Find the value of E in terms of W .

In this case the numbers E are so small that it is well to take the unit of length on the E -axis equal to ten times the unit on the W -axis in plotting the corresponding points.

When this is done the graph is that in the adjacent figure. It is seen that the points lie almost on a straight line and that the line comes very near to passing through the origin. This suggests that the relation between E and W be taken in the

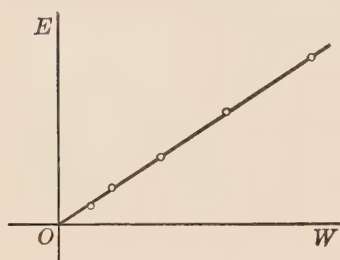


FIG. 111

form $E = mW$,

where m is a constant to be determined. Substituting for (W, E) the pair $(14, 1.79)$, we have for m approximately the value .128. Hence the required equation is

$$E = .128 W.$$

By comparing this with the given table of values it is seen to fit each pair (W, E) closely but not exactly, the errors being due primarily to errors in measurement. The equation $E = .128 W$ is taken as the mathematical expression of the law which is being sought.

In practice one chooses two points on the line so that the line when drawn will pass near to all the points. If the resulting equation does not fit the data well, one may choose other points through which to pass the line and so bring the equation into closer agreement with the data. It is usual to take for these two points two of those supplied by the table of data; but there are cases in which a more closely fitting line can be obtained by taking a line through two points near two of the points supplied by the original data. It is preferable that the two points used be well separated from each other.

EXERCISES

1. It is found that the number W of grams of potassium iodide which will dissolve in 100 grams of water at temperature T is given by the following table:

$T = 10$	15	20	25	30	40	50
$W = 136$	140	144	148	152	160	168

Find the value of W in terms of T and check the result.

2. The cost C of printing a certain pamphlet varies with the number N of copies in accordance with the following table:

$N = 1000$	2000	3000	4000	5000
$C = 100$	107.50	115	122.50	130

Find the formula for C in terms of N and check the result.

3. The latent heat L of steam in calories is given by the following table for various values of the temperature T :

$T = 70$	80	90	100	110	120	130
$L = 557$	550	543	536	530	523	515

Find the formula for L in terms of T and check the result.

4. Two variables u and v have corresponding values as in the following table:

$u = 1$	2	3	4	5	6	7	8
$v = .14$	.26	.38	.51	.63	.76	.89	1.01

Find the relation connecting u and v .

119. **The power law.** There are many phenomena which are connected by a law of the form

$$y = kx^n. \quad (1)$$

This is known as the *power law*. If we take the logarithms of both members of this equation, we have

$$\log y = n \log x + \log k.$$

If now we put

$$u = \log y, \quad v = \log x, \quad c = \log k, \quad (2)$$

the last equation becomes

$$u = nv + c. \quad (3)$$

Since this equation is linear its graph will be a straight line if the values of v are taken for abscissas and the corresponding values of u for ordinates. From this result we have the following method for determining whether a given table of corresponding values of y and x can be expressed by means of the power law: Form the corresponding table of values for $\log y$ and $\log x$, and call these values respectively u and v ; if the graph of the points (u, v) is linear, then y and x are connected as in the power law. The method of using this rule will now be illustrated by means of an example.

EXAMPLE. Let s be the number of feet passed over in t seconds by a body falling from rest. Then we have the following table of corresponding values :

$t =$	1	2	3	4	5
$s =$	16.08	64.32	144.72	257.28	402

Find the equation connecting s and t .

We form the following table of values of logarithms (base 10 being employed) :

$u = \log s =$	1.2063	1.8083	2.1606	2.4104	2.6042
$v = \log t =$	0	.3010	.4771	.6021	.6990

The points (u, v) lie very nearly on a straight line, as the student should show by plotting them. Hence we take

$$u = nv + c$$

as the relation between u and v . Using the first pair of values (u, v) in the table, we have $c = 1.2063$. Then using the last pair of values, we have

$$2.6042 = .6990 n + 1.2063,$$

whence it follows that $n = 2$ (with a very close approximation). From equations (2) and the value just determined for c we have

$$\log k = c = 1.2063.$$

Then from a table of logarithms we find that $k = 16.08$. Substituting these values for n and k in the equation

$$s = kt^n,$$

we have for the required relation

$$s = 16.08 t^2.$$

The student should check this with the table of values given in the problem.

EXERCISES

1. On measuring the pressures P and corresponding volumes V of a certain quantity of gas kept at a constant temperature in a cylinder with a movable piston, the following table of corresponding values was obtained :

$P =$	100	50	33.33	25	20	16.67	14.29
$V =$	1	2	3	4	5	6	7

Assuming a relation of the form $P = kV^n$, show that $n = -1$ and thence obtain the relation $PV = 100$ connecting P and V .

2. Let F denote the attraction of a certain magnet for a given piece of iron when they are a distance d apart. By experiment it is found that F and d have the following corresponding values :

$d = 1$	2	3	4	5	7	10
$F = 1$	.25	.11	.06	.04	.02	.01

Find the law expressing F in terms of d and check the result.

3. A body falls from rest. Let v be the speed which it has attained after falling h feet. The following table of corresponding values is determined by experiment :

$h =$	1	4	9	16	25	36
$v =$	8.02	16.04	24.06	32.08	40.10	48.12

Find the expression for v in terms of h and check the result.

4. If we take the distance of the earth from the sun as the unit of length, the distances D of the planets from the sun and their periods T (in years) of revolution about the sun are given by the following table :

	MERC.	VENUS	EARTH	MARS	JUP.	SAT.	URANUS	NEP.
$D =$	.387	.723	1.00	1.53	5.20	9.54	19.2	30.1
$T =$	.241	.615	1.00	1.88	11.9	29.5	83.7	165

Determine the relation between D and T and thus prove Kepler's third law, namely, that the squares of the times of revolution of the planets about the sun are proportional to the cubes of their distances from the sun.

120. The compound interest law. There is a considerable variety of empirical facts giving rise to a relation between the variables y and x which has the form

$$y = ab^x, \quad (4)$$

where a and b are constants. This is known as the *compound interest law*, owing to the similarity of equation (4) to the equation which expresses the amount of a given sum of money at compound interest. Thus if a sum p is put at compound interest at the rate r the amount a accumulated in n periods is given by the formula

$$a = p(1 + r)^n.$$

In order to find a method for testing whether a given set of data satisfy a law of type (4) we proceed as follows. Taking the logarithms of both members of equation (4), we have the relation

$$\log y = x \log b + \log a. \quad (5)$$

On putting $s = \log y$, $m = \log b$, $k = \log a$, (6)
equation (5) takes the form

$$s = mx + k. \quad (7)$$

If x is used as the abscissa and s as the ordinate, the graph of equation (7) is a straight line. Therefore the condition that y and x are connected by an equation of the form (4) is that the points (x, s) lie on a straight line, where $s = \log y$.

EXAMPLE. The number N of bacteria in a given culture t hours after they were first observed was found to be that given in the following table:

$t =$	0	1	2	3	4	5	6
$N =$	125	209	340	561	924	1525	2512

Find N in terms of t .

For t and $\log N$ we have the following table:

$t =$	0	1	2	3	4	5	6
$s = \log N =$	2.0969	2.3201	2.5315	2.7490	2.9657	3.1833	3.4000

By plotting the points (t, s) it is found that they lie approximately on a straight line. Hence we have to determine m and k in the relation

$$s = mt + k.$$

From the first pair (t, s) we see that $k = 2.0969$. Then from the last pair we have $m = 0.2172$. Comparing with equations (6), we have

$$\log b = m = 0.2172, \quad \log a = k = 2.0969.$$

From a table of logarithms we then have

$$b = 1.649, \quad a = 125.$$

Hence, on substituting in the equation $N = ab^t$, we have

$$N = 125(1.649)^t.$$

The student should check this with the original table of values.

EXERCISES

1. The speed v of a certain chemical reaction is related to the temperature T in accordance with the following table:

$T = 0$	1	2	3	4	5
$v = 5$	7.5	11.3	16.9	25.3	38.0

Find the expression for v in terms of T .

2. The atmospheric pressure p (in inches) at various heights h was found by observation to give rise to the following table (the unit for h being 1000 feet):

$h = 0$	5	10	15	20	25	30
$p = 30$	25	20	17	14	11.5	9.5

Find the value of p in terms of h and check the result.

3. The rate R of rotation of a wheel under water t seconds after the power was cut off was found to be in accordance with the following table:

$t = 0$	10	20	30	40
$R = 1500$	674	303	136	61

Find the value of R in terms of t and check the result with the table.

121. **Some special empirical laws.** There are several special forms of equations which may easily be adjusted to fit suitable data.

Thus the equation of the hyperbola

$$xy = ax + by \quad (8)$$

may be transformed by division by x into the form

$$y = a + b \frac{y}{x}.$$

Then if we plot the values (s, y) , where $s = y/x$, we obtain a straight line graph. This is one test for determining whether the data can be described by a law of the form (8). A second test for the same result may be obtained by dividing equation (8) through by y , whence we obtain the equation

$$x = a \frac{x}{y} + b.$$

If then the points (x, t) , where $t = x/y$, lie on a straight line, the data can be represented by an equation of the form (8).

A third test may be obtained by dividing the equation through by xy ; whence we have

$$1 = \frac{a}{y} + \frac{b}{x}.$$

Then if the points (s, t) , where $s = 1/x$, $t = 1/y$, lie on a straight line, the data can be represented by an equation of the form (8). Each of the three methods will lead to the same values of a and b and hence to the same equation (8).

Again, a relation of the form

$$y = a + \frac{b}{x^2} \quad (9)$$

will lead to a straight line graph for the points (s, y) where $s = 1/x^2$. When this condition is satisfied the values of a and b can be determined so that the data will be represented by equation (9).

From the equation
$$y = \frac{ax}{1 + bx} \quad (10)$$

we have, on clearing of fractions, transposing, and dividing by b , the equation

$$xy = \frac{a}{b}x - \frac{1}{b}y.$$

Hence equation (10) comes under the case of equation (8).

Finally, it is sometimes desirable to try an equation of the form

$$y = a + bx + cx^2 \quad (11)$$

or even of the more general form

$$y = a + bx + cx^2 + \cdots + kx^n. \quad (12)$$

If an equation of one of the latter forms is suspected one substitutes pairs of values (x, y) until he has as many equations as unknown coefficients and then computes these coefficients. The result should then be checked with the remaining pairs (x, y) .

Various other forms of equations may be needed for other types of problems. Those which we have given include the most frequently occurring ones.

The following exercises involve problems of all the types discussed in the chapter.

EXERCISES

1. In studying the deflection of beams of a given cross-section and varying length the following deflections were produced by a given load W :

Length (in.) =	12	16	20	24	32	40
Deflection (in.) =	.034	.086	.170	.290	.684	1.42

Find the formula for the deflection d in terms of the length l .

2. Obtain an equation to fit the following values of p and v :

$v =$	8.4	9.4	10	11	12.4	14	16	18
$p =$	52.5	46.0	43.0	39.0	34.0	30.0	26.5	23.0

3. Obtain an equation to fit the following values of u and v :

$u =$	2.0	4.4	6.8	9.2	20.4	25.6
$v =$	6.80	2.00	1.19	.92	.67	.64

4. Adjust the compound interest law to fit the following table of values:

$x =$	1.1	1.2	1.4	1.6	1.8	1.9	2.0
$y =$	6.01	6.64	8.11	9.91	12.10	13.37	14.78

5. Obtain an equation to fit the following table of values:

$x =$	1	2	3	4.5	6	10
$y =$	.93	1.13	1.33	1.63	1.93	2.73

6. Obtain an equation to fit the following table:

$x =$	2	3	4	5	6	7	8
$y =$	2.00	1.50	1.33	1.25	1.20	1.17	1.14

7. By means of an equation of the form $y = a + bx + cx^2$ fit the following data:

$x =$	1	2	3	4	5	6	7	8
$y =$	6	11	18	27	38	51	66	83

8. By means of an equation of the form $y = a + bx + cx^2 + dx^3$ fit the following data:

$x =$	1.1	1.2	1.3	1.4	1.5	1.8	2.0
$y =$	2.54	2.17	3.89	4.70	5.62	9.07	12.00

9. By means of an equation of the form

$$y = \frac{ax}{1 + bx}$$

fit the following data:

$x =$	1	2	3	4	5	6
$y =$	.91	1.70	2.30	2.90	3.30	3.70

10. By means of an equation of the form

$$y = a + \frac{b}{x^2}$$

fit the following data :

$x =$	.5	1.0	1.5	2.0	2.5	3.0	4.0
$y =$	9.0	3.0	1.9	1.5	1.3	1.2	1.1

Check the result obtained and draw a graph.

11. By means of an equation of the form

$$y = \frac{ax}{1 + bx}$$

fit the following data :

$x =$	0	1	2	3	4	5	7	10
$y =$	0	1	1.3	1.5	1.6	1.7	1.8	1.8

Check the result obtained and draw a graph.

12. By means of an equation of the form

$$xy = ax + by$$

fit the following data :

$x =$	3	4	5	6	7	8	9
$y =$	9	6	5	4.5	4.2	4	3.85

Check the result obtained and draw a graph.

SOLID ANALYTIC GEOMETRY

CHAPTER XI

POINTS, PLANES, AND LINES

122. Rectangular coördinates. In a space of three dimensions a point can be definitely located by giving its directed distances from three mutually perpendicular planes, called **coördinate planes**. In the figure the three planes XOY , YOZ , and XOZ are called the **xy -**, **yz -**, and **xz -planes**, respectively, and their lines of intersection OX , OY , and OZ are called the x -, y -, and z -axes, respectively. The point O is the origin. The directed distances of a point P from the yz -plane, the xz -plane, and the xy -plane are called the x -, y -, and z -coördinates, respectively, of the point P . Thus the coördinates of $P(x, y, z)$ are represented by

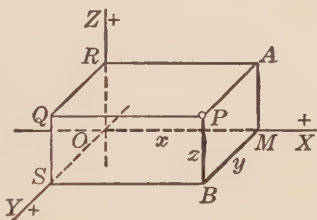


FIG. 112

$$x = OQ, \quad y = AP, \quad z = BP.$$

It is frequently convenient to locate P by drawing

$$x = OM, \quad y = MB, \quad z = BP.$$

The three coördinate planes divide space into eight divisions called octants. The octant in which the coördinates are all positive is called the first octant and the others are seldom numbered. They are distinguished by the signs of the coördinates.

123. Distance between two points. Drawing lines through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ parallel to each of the coördinate axes, and other lines as indicated in the figure, we have a rectangular parallelepiped of which the distance between P_1 and P_2 is a diagonal. The length of the diagonal in terms of the sides is

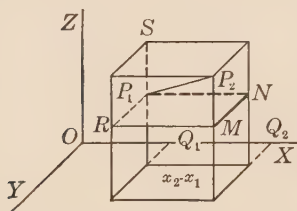


FIG. 113

$$P_1P_2 = \sqrt{P_1N^2 + NM^2 + MP_2^2}.$$

But $P_1N = Q_1Q_2 = x_2 - x_1$, and similarly, $NM = y_2 - y_1$ and $MP_2 = z_2 - z_1$. Hence

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

In particular, the distance from the origin $(0, 0, 0)$ to a point $P_1(x_1, y_1, z_1)$ is

$$OP_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}.$$

124. Direction of a line. The direction of a line through the origin is determined if we know the angles it makes with the positive directions of the three coördinate axes. These angles are called the **direction angles** of the line and are denoted by α , β , and γ as the angles made with the x -, y -, and z -axes, respectively. It is found that the cosines of the angles are more useful than the angles, and the direction of a line is usually given by the cosines of its direction angles. The cosines of the direction angles are called the **direction cosines** of the line. If the direction cosines of a line AB are $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, the direction cosines of the line BA are $\cos(180^\circ - \alpha)$, $\cos(180^\circ - \beta)$, and $\cos(180^\circ - \gamma)$.

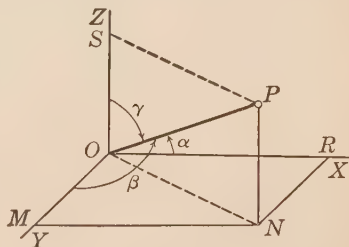


FIG. 114

The direction cosines of a line through the origin are related. In the figure we see that

$$OR = OP \cos \alpha, \quad RN = OM = OP \cos \beta, \quad NP = OS = OP \cos \gamma.$$

Squaring and adding, we have

$$OR^2 + RN^2 + NP^2 = OP^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma).$$

But $OR^2 + RN^2 + NP^2 = OP^2$;

hence $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

The direction cosines of a line not through the origin are defined to be the same as the direction cosines of a line parallel to the given line and passing through the origin. Thus in Fig. 113 we have

$$\cos \alpha = \frac{P_1N}{P_1P_2}, \quad \cos \beta = \frac{P_1R}{P_1P_2}, \quad \cos \gamma = \frac{P_1S}{P_1P_2},$$

or $\cos \alpha = \frac{x_2 - x_1}{P_1P_2}, \quad \cos \beta = \frac{y_2 - y_1}{P_1P_2}, \quad \cos \gamma = \frac{z_2 - z_1}{P_1P_2}.$

Squaring and adding, we have

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}{P_1P_2^2} = 1.$$

This verifies the fact that for any line the sum of the squares of the direction cosines is 1.

Any three numbers a , b , and c (not all zero) are proportional to the direction cosines of some line. Let (a, b, c) be any point P (Fig. 114). We have

$$OP = \sqrt{a^2 + b^2 + c^2},$$

and $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad \cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}},$

$$\cos \gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

Hence a , b , and c are proportional to the direction cosines of the line OP , or of any line parallel to OP . The direction cosines of the line joining the origin to any point (x, y, z) are

$$\cos \alpha = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad \cos \beta = \frac{y}{\sqrt{x^2 + y^2 + z^2}},$$

$$\cos \gamma = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

Numbers proportional to the direction cosines of a line are called **direction numbers** of the line. The direction cosines may be obtained from the direction numbers by dividing each number by the square root of the sum of the squares of the three direction numbers.

EXAMPLE. Find the distance between the points $(9, 2, -1)$ and $(-3, 5, -5)$ and the direction of the line through them.

The distance between the points is

$$d = \sqrt{[9 - (-3)]^2 + (2 - 5)^2 + [-1 - (-5)]^2} = \sqrt{169} = 13.$$

Considering the direction of the line as that from the first point to the second, the direction cosines are

$$\begin{aligned}\cos \alpha &= \frac{-3 - 9}{13} = -\frac{12}{13}, & \cos \beta &= \frac{5 - 2}{13} = \frac{3}{13}, \\ \cos \gamma &= \frac{-5 - (-1)}{13} = -\frac{4}{13}.\end{aligned}$$

If we consider the direction of the line as that from the second point to the first, the direction cosines are

$$\cos \alpha = \frac{12}{13}, \quad \cos \beta = -\frac{3}{13}, \quad \cos \gamma = \frac{4}{13}.$$

The numbers 12, -3 , and 4, or any multiple of these numbers, will serve as direction numbers of the line through the two points.

125. Angle between two lines. Let $P(x, y, z)$ and $P_1(x_1, y_1, z_1)$ be two points and let the direction cosines of OP and OP_1 be $\cos \alpha, \cos \beta,$ and $\cos \gamma$, and $\cos \alpha_1, \cos \beta_1,$ and $\cos \gamma_1$, respectively. In the triangle OPP_1 we have

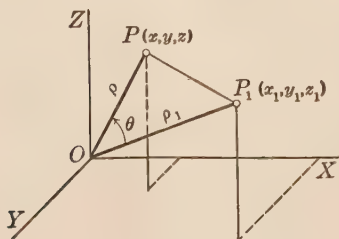


FIG. 115

$$\begin{aligned}\overline{PP_1}^2 &= \rho^2 + \rho_1^2 - 2\rho\rho_1\cos\theta, \\ \text{or } \cos\theta &= \frac{\rho^2 + \rho_1^2 - \overline{PP_1}^2}{2\rho\rho_1}.\end{aligned}$$

But (Sec. 123) $\rho^2 = x^2 + y^2 + z^2$, $\rho_1^2 = x_1^2 + y_1^2 + z_1^2$, and $\overline{PP_1}^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$.

Making these substitutions and reducing, we have

$$\cos \theta = \frac{xx_1 + yy_1 + zz_1}{\sqrt{x^2 + y^2 + z^2} \cdot \sqrt{x_1^2 + y_1^2 + z_1^2}}.$$

The direction cosines of OP are

$$\cos \alpha = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad \cos \beta = \frac{y}{\sqrt{x^2 + y^2 + z^2}},$$

and
$$\cos \gamma = \frac{z}{\sqrt{x^2 + y^2 + z^2}},$$

and the direction cosines of OP_1 are

$$\cos \alpha_1 = \frac{x_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}, \quad \cos \beta_1 = \frac{y_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}},$$

and
$$\cos \gamma_1 = \frac{z_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}.$$

Hence we may write

$$\cos \theta = \cos \alpha \cos \alpha_1 + \cos \beta \cos \beta_1 + \cos \gamma \cos \gamma_1. \quad (1)$$

If the lines do not pass through the origin or do not intersect, the angle between them is defined to be the same as the angle between lines through the origin parallel to the given lines. Hence the cosine of the angle between any two lines is given in terms of the direction cosines of the two lines by equation (1).

The condition that the two lines are perpendicular is that $\cos \theta = 0$, that is,

$$\cos \alpha \cos \alpha_1 + \cos \beta \cos \beta_1 + \cos \gamma \cos \gamma_1 = 0.$$

EXAMPLE. Find the angle between the line joining the origin to (3, 4, 5) and the line joining the points (2, 1, 4) and (10, 5, 5).

The direction cosines of the first line are $\frac{3}{5\sqrt{2}}$, $\frac{4}{5\sqrt{2}}$, and $\frac{5}{5\sqrt{2}}$, and the direction cosines of the second line are $\frac{8}{9}$, $\frac{4}{9}$, and $\frac{1}{9}$. Hence

$$\cos \theta = \frac{8 \cdot 3 + 4 \cdot 4 + 1 \cdot 5}{9 \cdot 5\sqrt{2}} = \frac{\sqrt{2}}{2},$$

and the angle between the lines is 45° .

EXERCISES

1. Find the length and direction cosines of the line segment drawn from the origin to each of the following points. Draw a figure in each case.

- (a) (4, 4, 2). (c) (-3, -2, 6). (e) (0, 5, 0).
 (b) (8, 4, -1). (d) (4, 2, 0). (f) (4, 0, 3).

2. Find the length and direction cosines of the line segments joining the following pairs of points:

- (a) (-2, 2, 1) and (4, 5, -1).
 (b) (5, -2, 3) and (-3, 6, -1).
 (c) (4, 5, 7) and (-2, 3, 7).
 (d) (10, 9, -2) and (3, 4, -3).
 (e) (3, 1, 2) and (-1, -3, 2).
 (f) (-3, 4, 5) and (7, 4, 5).

3. (a) If $\alpha = \beta = 60^\circ$, find γ (two solutions). (b) If $\alpha = \beta = 30^\circ$, find γ .

4. Draw the line through the origin which has $\cos \alpha = \cos \beta$
 $\underline{\underline{=}} \sqrt{2}/2$.

5. Find the direction angles of a line which makes equal angles with the coordinate axes.

6. Find the direction cosines of the lines whose direction numbers are

- (a) 1, 2, and 2. (c) 5, 6, and -8. (e) 2, 3, and 6.
 (b) 2, -6, and 9. (d) 1, 1, and 0.

7. Show that the points (1, -2, 3), (-4, -6, 1), and (11, 6, 7) lie on a straight line.

8. Find the angle between the line through (5, -1, 7) and (1, -3, 3) and a line having $\cos \alpha = \frac{1}{15}$, $\cos \beta = \frac{2}{3}$, and $\cos \gamma = \frac{2}{15}$.

9. Find the angle between the line through (2, 3, -2) and (-5, -2, -6) and a line whose direction numbers are 2, -1, and 0.

10. Show that the line through (1, 5, 8) and (3, 2, 4) is (a) parallel to the line through (0, 0, 0) and (-4, 6, 8); (b) perpendicular to the line through (-1, 9, -2) and (-7, 1, 1).

11. Show in two ways that (a) (5, 1, 5), (4, 3, 2), and (-3, -2, 1) are the vertices of a right triangle; (b) (3, 7, -2), (-1, 8, 3), and (-3, 4, -2) are the vertices of an isosceles

triangle; (c) $(4, 2, 4)$, $(10, 2, -2)$, and $(2, 0, -4)$ are the vertices of an equilateral triangle.

12. Show that the midpoint of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

126. **Equation of a plane.** Let ON be the line from the origin perpendicular to the plane, and let its length be p and its direction cosines be $\cos \alpha_1$, $\cos \beta_1$, and $\cos \gamma_1$. Let $P(x, y, z)$ be any other point in the plane, and let the direction cosines of OP be $\cos \alpha$, $\cos \beta$, and $\cos \gamma$. The triangle ONP is a right triangle for all points P which lie in the plane and for no other points. We have

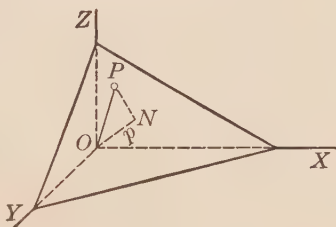


FIG. 116

$$p = ON = OP \cos \theta. \quad (\theta = \angle NOP)$$

But $\cos \theta = \cos \alpha \cos \alpha_1 + \cos \beta \cos \beta_1 + \cos \gamma \cos \gamma_1$, $\cos \alpha = x/OP$, $\cos \beta = y/OP$, and $\cos \gamma = z/OP$. Making these substitutions and reducing, we have

$$x \cos \alpha_1 + y \cos \beta_1 + z \cos \gamma_1 = p \quad (2)$$

as the equation of a plane, in which the constants are the length and the direction cosines of the perpendicular from the origin to the plane. The length p is positive when measured from the origin to the plane.

Conversely, any equation of the form of equation (2) represents a plane.

By retracing the steps by which equation (2) is obtained, we see that equation (2) is satisfied by points P for which the angle ONP is a right angle, and by no other points. All such points lie in a plane. Hence any equation in the form of (2) represents a plane.

127. Theorem. *The equation*

$$Ax + By + Cz + D = 0 \quad (3)$$

represents a plane, provided that A , B , and C are not all zero.

Transposing D and dividing by $\pm\sqrt{A^2 + B^2 + C^2}$, we have

$$\frac{A}{\pm\sqrt{A^2 + B^2 + C^2}}x + \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}}y + \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}}z = \frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}}. \quad (4)$$

By Sec. 124, the quantities

$$\frac{A}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad \text{and} \quad \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}}$$

are the direction cosines of a line. Equation (4) is, therefore, in the form of equation (2), and hence equation (3) represents a plane perpendicular to the line whose direction cosines are

$$\cos \alpha_1 = \frac{A}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad \cos \beta_1 = \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}}, \\ \cos \gamma_1 = \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

The distance p from the origin to the plane measured along the normal is

$$p = \frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}},$$

where the sign before the radical is chosen so that p is positive.

Planes parallel to coördinate axes or parallel to coördinate planes have equations which are special cases of equation (3).

In case a plane is parallel to a coördinate axis, say the x -axis, the direction angle made by the normal with that axis is 90° and the corresponding direction cosine, $\cos \alpha_1$, is zero. Hence, in equation (3), $A = 0$. Similarly, the equation of a plane parallel to the y -axis has no y -term, and a plane parallel to the z -axis has no z -term. Thus the equations

$$2y - 3z = 5, \quad x + 7z - 2 = 0, \quad 3x + 2y + 4 = 0$$

represent planes parallel to the x -, y -, and z -axes, respectively.

In case a plane is parallel to a coördinate plane, the normal is at right angles to the two axes lying in that coördinate plane. Hence two direction cosines are zero. The only variable in the equation of such a plane is the one corresponding

to that axis to which the given plane is perpendicular. Thus the equations $2x = 7$, $y + 3 = 0$, $z = 9$ represent planes parallel to the yz -, the xz -, and the xy -planes, respectively.

EXAMPLE. Discuss the locus of the equation

$$2x + 3y + 4z = 12.$$

The equation represents a plane since it is of first degree. Direction numbers of the normal to the plane are 2, 3, and 4. The direction cosines are found from the direction numbers to be $\cos \alpha_1 = 2/\sqrt{29}$, $\cos \beta_1 = 3/\sqrt{29}$, and $\cos \gamma_1 = 4/\sqrt{29}$. The distance from the origin to the plane is $12/\sqrt{29}$. The intercepts on the x -, y -, and z -axes are 6, 4, and 3, respectively.

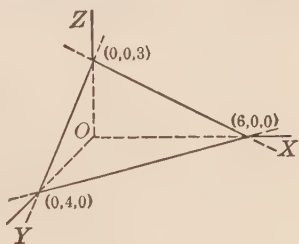


FIG. 117

The lines in which the plane intersects the coordinate planes are called the traces of the plane. In the xy -plane, $z = 0$. Hence, to find the equation of the trace in the xy -plane set $z = 0$ in the equation of the plane. Similarly, to find the xz -trace set $y = 0$, and to find the yz -trace set $x = 0$. The equations of the xy -, yz -, and xz -traces of the plane under discussion are

$$2x + 3y = 12, \quad 3y + 4z = 12, \quad x + 2z = 6,$$

respectively. The traces and intercepts are shown in Fig. 117.

128. To find the equation of a plane satisfying given conditions. Planes are usually determined by three given points or by the direction of the normal and one point.

In case three points are given, substitute their coordinates in turn in equation (3). We thus obtain three equations from which three of the numbers A , B , C , and D may be determined in terms of the fourth. Or, by dividing each of the equations by one of the numbers, say A , we have three equations in the unknowns B/A , C/A , and D/A which may be solved simultaneously.

EXAMPLE 1. Find the equation of the plane through the points (1, 1, 1), (-2, -1, 2), and (3, 9, 3).

Equation (3) may be divided by D^* and written

$$A'x + B'y + C'z + 1 = 0,$$

where $A' = A/D$, $B' = B/D$, and $C' = C/D$. The equations expressing the condition that the points are in the plane are

$$A' + B' + C' + 1 = 0,$$

$$-2A' - B' + 2C' + 1 = 0,$$

$$3A' + 9B' + 3C' + 1 = 0.$$

Solving simultaneously, we have $A' = -\frac{1}{2}$, $B' = \frac{1}{3}$, and $C' = -\frac{5}{6}$. Substituting these values and reducing, we have the desired equation,

$$3x - 2y + 5z - 6 = 0.$$

In case the direction of the normal and one point are given, we may use equation (3). The numbers A , B , and C are direction numbers of the normal to the plane. From the given direction of the normal we obtain direction numbers which are used for A , B , and C ; and D is determined so that the plane passes through the given point.

EXAMPLE 2. Find the equation of the plane through the point (3, 5, -1) perpendicular to the line through (-1, 3, 7) and (4, 2, 5).

Direction numbers of the line through the given points are 5, -1, and -2. Hence we may take $A = 5$, $B = -1$, $C = -2$, and write

$$5x - y - 2z + D = 0$$

as the equation of a plane perpendicular to the given line. Substituting the coördinates of the point (3, 5, -1), we find the value of D , $D = -12$, for which the plane passes through the given point. The required equation is, therefore,

$$5x - y - 2z - 12 = 0.$$

* Division must be by a quantity not zero. In case $D = 0$, it will be impossible to solve the set of conditional equations. In such a case, we should have to divide by one of the other numbers.

EXERCISES

1. Find the intercepts on the axes and the traces in the coördinate planes of each of the following planes, and sketch the portion of each plane which is intercepted by the coördinate planes:

(a) $2x + 3y + 6z = 14$.

(c) $11x + 10y - 2z = 20$.

(b) $x - y - 5z + 9 = 0$.

(d) $4x - y + z = 8$.

2. Find the distance from the origin to each of the planes in Exercise 1.

3. Sketch the following planes:

(a) $x + 2y = 4$.

(e) $x + y = 0$.

(i) $y = -3$.

(b) $2y - z = 6$.

(f) $2z = x$.

(j) $z = k$.

(c) $2x - 3y = 8$.

(g) $z = ky$.

(k) $z^2 = 4$.

(d) $x - y = 0$.

(h) $x = 5$.

(l) $x^2 - 2x - 3 = 0$.

4. Picture the volume in the first octant bounded by the planes $x + y + z = 6$, $2y = x$, $y = 2x$, $z = 3$, and the xy -plane.

5. Picture the volume in the first octant bounded by the planes $z = x + y$, $x + y = 4$, and the coördinate planes.

6. Find the equation of the plane containing the z -axis and the point $(2, 3, 0)$.

7. Find the equation of the plane parallel to the x -axis whose intercepts on the y - and z -axes are 3 and 5, respectively.

8. Write the equation of the plane parallel to the y -axis whose trace in the xz -plane contains the points $(2, 0, 9)$ and $(6, 0, 1)$.

9. Find the equation of the plane determined by the following points: (a) $(3, 3, 2)$, $(1, -1, -5)$, and $(-7, 1, 1)$; (b) $(0, 0, 0)$, $(2, 6, 3)$, and $(5, -7, 2)$.

10. Show that the equation of a plane whose x -, y -, and z -intercepts are a , b , and c , respectively, is $x/a + y/b + z/c = 1$.

11. Find the direction cosines of a line perpendicular to each of the following planes:

(a) $2x - y + 3z = 5$.

(c) $4x - 3y = 0$.

(b) $6x - 2y + 9z + 7 = 0$.

(d) $z = 2$.

12. Find the equation of the plane through $(3, 6, 2)$ perpendicular to the line joining that point to the origin.

13. Write the equation of the plane bisecting at right angles the line segment joining $(3, 6, -2)$ and $(5, -2, 4)$.

14. Write the equation of the plane through $(8, 3, 6)$ parallel to the plane $x - 3y + 2z = 1$.

15. If θ is the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, show that

$$\cos \theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(Consider the angle between two lines perpendicular to the planes, respectively.)

16. Show that the plane $2x + 2y + z + 5 = 0$ intersects the plane $5x + 4y - 3z - 2 = 0$ at an angle of 45° .

17. Find the angle between the planes $2x + y - z + 8 = 0$ and $x + 2y + z - 4 = 0$.

18. Determine the angle which the plane $x + y + z - d = 0$ makes with each of the coordinate planes.

19. What is the condition that the planes in Exercise 15 be (a) perpendicular? (b) parallel? (c) coincident?

20. Show that the plane $9x - 6y + 2z - 11 = 0$ is parallel to the plane $18x - 12y + 4z - 55 = 0$. Find the distance between the two planes.

21. Show that the plane $3x - y + 2z = 6$ is perpendicular to the plane $6x + 4y - 7z + 2 = 0$.

22. Find the point of intersection of the following planes: $x + 2y + z - 6 = 0$, $3x - y - 2z + 5 = 0$, $4x + 3y - z + 1 = 0$.

23. Show analytically that the locus of points equidistant from $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is a plane which bisects at right angles the line segment joining P_1 and P_2 .

24. Find the point equidistant from $(0, 0, 0)$, $(2, 1, -5)$, $(1, 0, -3)$, and $(-1, 4, 1)$.

129. Equations of a straight line. A straight line is determined by the intersection of two planes. An indefinite number of planes pass through any straight line. The equations of any two of the planes through the line, considered as simultaneous equations, represent the line. It is usually convenient to represent a line by planes which are parallel to coordinate axes. The equations representing such planes have but two variables each.

130. Projection planes. A plane parallel to the z -axis and passing through a given line is perpendicular to the xy -plane and cuts that plane in a line which is the projection of the given line upon the xy -plane. Similarly, planes through the given line and parallel to the y -axis and to the x -axis, respectively, cut the xz -plane and the yz -plane in lines which are the projections of the given line upon those planes. Such planes are called **projection planes** of the line. Each line not parallel to an axis has three projection planes and any two of them determine the line. The method of finding the projection planes of a line whose equations are in the general form is illustrated in the following example.

EXAMPLE. Find the equations of the projection planes of the line whose equations are

$$2x + 2y + 3z = 23$$

$$\text{and } 4x + 3y + 9z = 52.$$

Eliminating z , y , and x in turn from the given equations, we obtain

$$2x + 3y = 17, \quad 2x + 9z = 35,$$

$$\text{and } y - 3z + 6 = 0$$

as the three required equations.

They represent the planes which project the given line upon the xy -, xz -, and yz -planes, respectively.

The segment AB in the figure represents a part of the given line, and AM , BN , and KL are segments of the traces of the three projecting planes in the coordinate planes.

131. Equations of a line through a point and in a given direction. Let (x_1, y_1, z_1) be the given point and let the direction cosines of the line be $\cos \alpha$, $\cos \beta$, and $\cos \gamma$. Let (x, y, z) be any other point on the line and call d the distance from (x_1, y_1, z_1) to (x, y, z) . Then

$$d = \frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma}.$$

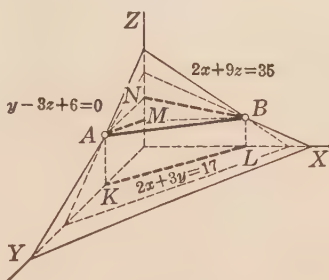


FIG. 118

These equalities exist for every d ; hence the equations of the line through the point (x_1, y_1, z_1) and with direction cosines $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are

$$\frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma}. \quad (5)$$

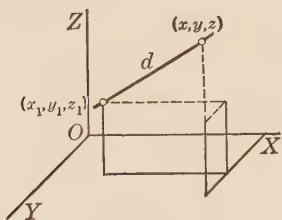


FIG. 119

Since equations (5) may be multiplied by the same constant without destroying the equality, it is not necessary that the denominators be the direction cosines of the line. It

is sufficient that the denominators be proportional to the direction cosines. Thus the equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad (6)$$

are the equations of a line through the point (x_1, y_1, z_1) and with direction numbers a , b , and c . Equations (6) may be written in the form of equations (5) by dividing each denominator by $\pm \sqrt{a^2 + b^2 + c^2}$.

Equations (5) or (6) are frequently called the **symmetrical** equations of a straight line. Either (5) or (6) are the equations of the projection planes of the line.

EXAMPLE. Find the equations of the line through the point $(2, 1, -3)$ and parallel to the line through the points $(5, 2, 3)$ and $(0, 6, 2)$.

The direction cosines are proportional to the differences of corresponding coördinates, hence the direction cosines of the line through $(5, 2, 3)$ and $(0, 6, 2)$ are proportional to 5 , -4 , and 1 . The required equations are, therefore,

$$\frac{x - 2}{5} = \frac{y - 1}{-4} = \frac{z + 3}{1}.$$

132. Equations of a line through two given points. Let the given points be $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. The direction cosines are proportional to $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$. The

equations of the line through P_1 and P_2 are, therefore, from equations (6) of the last section,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

133. Transformations of the general equations of a line to type forms. The following example will illustrate the method of transformation of the general equations of a line into type forms from which we can readily obtain certain properties of the line.

EXAMPLE. Transform the equations of the line

$$2x + 2y + 3z = 23 \quad \text{and} \quad 4x + 3y + 9z = 52,$$

in the example of Sec. 130, into type forms and find its direction cosines.

In that example we found the projection planes of the given line to be

$$2x + 3y = 17, \quad 2x + 9z = 35, \quad \text{and} \quad y - 3z + 6 = 0. \quad (7)$$

Any two of these planes determine the line. Solving the first two equations of (7) for $2x$, we may write

$$2x = -3y + 17 = -9z + 35.$$

Dividing by 18, and writing each variable with a positive sign, we have

$$\frac{x}{9} = \frac{y - \frac{17}{3}}{-6} = \frac{z - \frac{35}{9}}{-2}. \quad (8)$$

These are symmetric equations of a line through the point $(0, \frac{17}{3}, \frac{35}{9})$ and with direction numbers 9, -6, and -2.

We may obtain the symmetric equations of the line with direction cosines instead of direction numbers by dividing each denominator of equations (8) by 11. We thus obtain

$$\frac{x}{\frac{9}{11}} = \frac{y - \frac{17}{3}}{-\frac{6}{11}} = \frac{z - \frac{35}{9}}{-\frac{2}{11}}. \quad (9)$$

The direction cosines are $\cos \alpha = \frac{9}{11}$, $\cos \beta = -\frac{6}{11}$, and $\cos \gamma = -\frac{2}{11}$.

The same straight line is here represented in four different ways by the general equations in the example and by the type forms (7), (8), and (9).

EXERCISES

1. Find the projection planes of the following lines and sketch each figure :

(a) $x + 2y + 2z = 10$ and $2x + y - z = 12$.

(b) $x + y + z = 15$ and $3y + 2z = 18$.

(c) $x + y = z$ and $x = 5$.

(d) $x + y = z$ and $y = x$.

(e) $\frac{x-4}{2} = \frac{y-1}{-2} = \frac{z-3}{-1}$.

2. Find the points where each line of Exercise 1 pierces the coördinate planes.

3. Write the equations of each of the following lines :

(a) Through $(6, 2, 4)$ and with $\cos \alpha = \frac{8}{9}$, $\cos \beta = \frac{4}{9}$, $\cos \gamma = \frac{1}{9}$.

(b) Through $(-1, 3, -2)$ and with direction numbers $6, -3, 2$.

(c) Through $(0, -3, 8)$ and $(7, 2, 9)$.

(d) Through $(5, 2, 0)$ and parallel to the line joining the origin and $(8, 5, 1)$.

(e) Through $(3, 2, 0)$ and with $\cos \alpha = \frac{4}{5}$, $\cos \beta = \frac{3}{5}$.

4. Write the equations of the line through $(2, 6, 1)$ parallel to the line

$$\frac{x-3}{11} = \frac{y-1}{2} = \frac{z-2}{10}.$$

5. Write the equations of the line through the origin making equal angles with the positive directions of the coördinate axes.

6. Show that the line determined by the points $(3, 1, 0)$ and $(1, 4, -3)$ is perpendicular to the line

$$\frac{x}{3} = \frac{y-3}{8} = \frac{z+7}{6}.$$

7. Show that the line

$$\frac{x-4}{-3} = \frac{y-1}{3} = \frac{z+1}{2}$$

lies in the plane $5x - y + 9z = 10$.

8. Write the equations of the line through $(5, 4, 8)$ and perpendicular to the plane $2x + 3y + 6z = 18$.

9. Find the foot of the perpendicular dropped from the point $(-8, 5, 2)$ upon the plane $3x - 2y - z + 8 = 0$.

10. Find the length of the perpendicular dropped from the point $(6, 0, -\frac{7}{2})$ upon the plane $8x + 6y - 5z = 3$.

11. Find the point on the line

$$\frac{x-5}{2} = \frac{y-11}{6} = \frac{z-12}{9}$$

equidistant from $(4, -1, 2)$ and $(-2, 3, 6)$.

12. Find the symmetric equations of the line determined by the following planes:

(a) $x - 6y + 6z + 35 = 0$ and $2x - 2y - 3z = 0$.

(b) $2x - y - 2z = 13$ and $x - 2y + 2z + 7 = 0$.

13. Find the direction cosines of the line determined by the following planes:

(a) $4x - y - z = 12$ and $2x + y - 2z + 6 = 0$.

(b) $x - 2y - z + 2 = 0$ and $x - 5y - 5z + 4 = 0$.

14. Write the symmetric equations of the line through the point $(-5, 2, -3)$ and parallel to the line determined by the planes $8x - 7y - 7z = 15$ and $4x - y - 6z = 20$.

MISCELLANEOUS EXERCISES

1. Find the equation of the plane parallel to the y -axis and containing the points $(7, 2, 1)$ and $(3, -4, 3)$.

2. Find the equation of the plane through $(-2, 1, 4)$ perpendicular to the line through $(8, 7, -1)$ and $(5, 2, 0)$.

3. Find the point equidistant from $(6, 0, 0)$, $(0, 6, 0)$, and $(0, 0, 12)$ and lying in the plane of these points.

4. Show that the planes $2x + y + z - 3 = 0$, $5x - y - 2 = 0$, $x + 2y + 2z - 3 = 0$, and $3x - 3y - 4z - 2 = 0$ meet in a common point.

5. Show that $(1, 4, 1)$, $(-3, 1, 0)$, $(5, 0, 1)$, and $(5, 7, 2)$ lie in a common plane.

6. Find the equation of the plane through $(1, 1, 1)$ and $(2, -2, -1)$ perpendicular to the plane $x + 3y - 8z + 2 = 0$.

7. Find the equation of the plane through $(1, -3, 2)$ perpendicular to the planes $x - 2y - 3z = 6$ and $3x + 4y - 4z = 1$.

8. Show that the distance from the plane whose equation is $x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0$ to the point (x_1, y_1, z_1) is equal to $x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p$. (See the method for finding the distance from a line to a point, Sec. 26.)

9. Find the distance from the plane $8x + 4y + z - 16 = 0$ to
(a) $(5, 2, 4)$; (b) $(1, 1, 1)$; (c) $(3, -1, -4)$.

10. Show that the line

$$\frac{x-2}{5} = \frac{y-8}{-4} = \frac{z+1}{9}$$

is parallel to the plane $6x + 3y - 2z - 3 = 0$, and find the distance from the plane to the line.

11. Find the distance from $(6, 1, 2)$ to the line

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-4}{4}.$$

12. Determine the angle which the line

$$\frac{x-3}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

makes with the plane $2x + y + z = 7$.

13. Find the equation of the plane containing the line

$$\frac{x+3}{2} = \frac{y-2}{-2} = \frac{z+2}{3}$$

and the point $(1, -3, 6)$.

14. Show that the lines

$$\frac{x+1}{3} = \frac{y-6}{1} = \frac{z-3}{2} \quad \text{and} \quad \frac{x-6}{2} = \frac{y-11}{2} = \frac{z-3}{-1}$$

lie in a common plane, and find the equation of the plane.

15. Write the equations of the two lines through $(6, 4, 1)$ with $\cos \alpha = \frac{2}{3}$ and parallel to the plane $x + y - 4z - 12 = 0$.

16. Show that the equation

$$a_1x + b_1y + c_1z + d_1 + k(a_2x + b_2y + c_2z + d_2) = 0$$

(k any constant) represents a plane containing the line of intersection of the planes

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2z + d_2 = 0.$$

17. Write the equation of the plane containing the line of intersection of the planes $x - 6y + 2z + 1 = 0$ and $2x + y - 7z + 3 = 0$ and passing through $(2, -1, 3)$.

18. Write the equation of the plane containing the line of intersection of the planes $x + 2y - z - 8 = 0$ and $3x - y + 4z + 2 = 0$ and perpendicular to the first of these planes.

19. Show that the following planes meet in a common line:
 $x - 3y + 5z - 2 = 0$, $2x + y - 6z - 4 = 0$, $9x + y - 19z - 18 = 0$.

CHAPTER XII

SURFACES AND CURVES

134. Equation of a cylindrical surface with elements parallel to an axis. Let us consider an equation in two variables only, say in x and y ,

$$x^2 + y^2 = 25. \quad (1)$$

In the xy -plane the equation represents a circle of radius 5 and with center at the origin. Let $P(x, y, 0)$ be a point on the curve and draw through P a line AB parallel to the z -axis. The coördinates of any point on this line will satisfy equation (1) since each point on AB has the same x - and y -coördinates as P . Hence the coördinates of every point on the surface generated by moving P around the circle, keeping AB parallel to the z -axis, will satisfy equation (1). A surface which can be generated by moving a line parallel to a fixed position is called a **cylindrical surface**. Hence equation (1) represents a cylindrical surface with elements parallel to the z -axis and with any cross-section parallel to the xy -plane a circle of radius 5. The circle (1) is called a **directrix** of the cylindrical surface.

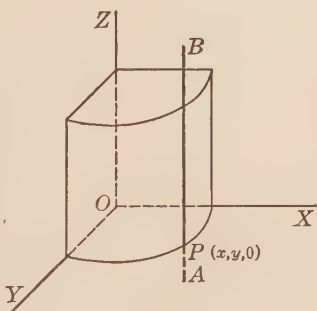


FIG. 120

In a similar manner it can be shown that any equation in two variables represents a cylindrical surface with elements parallel to the axis corresponding to the variable which is lacking in the equation. Cylindrical surfaces are frequently given names corresponding to the directrix, such as elliptic, parabolic, etc. A plane may be looked upon as a cylindrical surface whose directrix is a straight line.

EXERCISES

1. Discuss the following cylindrical surfaces. Sketch each surface. In each case what curve will serve as a directrix?

(a) $x^2 + z^2 = 16$.

(f) $y^2 = 16x$.

(b) $x^2 + y^2 - 2ax = 0$.

(g) $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$.

(c) $y^2 + z^2 + 4y - 6z = 0$.

(h) $xz = 6$.

(d) $2x - 3y = 6$.

(i) $y^2 = x^3$.

(e) $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

(j) $y = \sin x$.

2. Find the equation of the locus of points equidistant from the yz -plane and the line determined by the planes $x = 4$ and $y = 0$. Interpret the result.

3. Rotate the x - and y -axes through 45° , keeping the z -axis fixed, and show that $z^2 = 4 - x - y$ represents a cylindrical surface. What curve will serve as a directrix?

135. Surfaces of revolution. A surface formed by revolving a plane curve about some line in its plane has the property that cross-sections perpendicular to the axis of revolution are circles. The radius of the circle in each case is the distance from the axis to the point on the curve which is in the plane of the cross-section. This enables us to obtain the equation

of the surface when we know the equation of the generating curve, as is illustrated in the following examples.

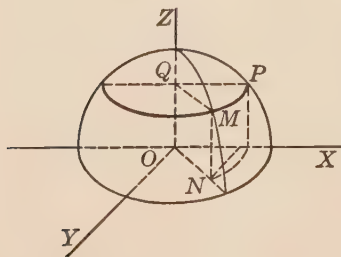


FIG. 121

EXAMPLE 1. The circle in the xz -plane whose equation is $x^2 + z^2 = a^2$ is revolved about the z -axis. Find the equation of the spherical surface thus generated.

Any point P on the given circle will generate a circle with center at Q and with radius QP . The plane of QMP is parallel to the xy -plane. For any point M we have

$$x^2 + y^2 = \overline{ON}^2 = \overline{QM}^2 = \overline{QP}^2.$$

But QP is the x -coordinate of the point P in the xz -plane. From the given equation we have

$$QP = x = \sqrt{a^2 - z^2}.$$

Hence the equation of the locus of P is

$$x^2 + y^2 = a^2 - z^2, \quad \text{or} \quad x^2 + y^2 + z^2 = a^2. \quad (2)$$

But P is any point on the circle, hence equation (2) represents the spherical surface formed by revolving the circle about the z -axis.

EXAMPLE 2. The line in the xy -plane whose equation is $x + y = 6$ is revolved about the x -axis. Find the equation of the conical surface thus generated.

Any point P on the line generates a circle whose plane is parallel to the yz -plane. For any point M on this circle

$$y^2 + z^2 = \overline{QM}^2 = \overline{QP}^2.$$

But in the xy -plane $QP = y$, hence $QP = y = 6 - x$.

The required equation of the conical surface is, therefore,

$$y^2 + z^2 = (6 - x)^2, \quad \text{or} \quad y^2 + z^2 - (6 - x)^2 = 0.$$

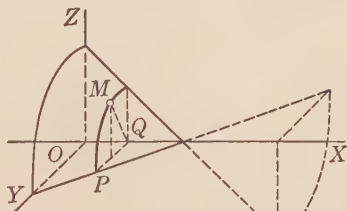


FIG. 122

EXERCISES

1. Find the equation of each of the following surfaces generated as indicated, and sketch each surface:

- (a) By revolving the parabola $z^2 = 4x$ about the x -axis.
- (b) By revolving the ellipse $\frac{x^2}{25} + \frac{z^2}{9} = 1$ about (1) the x -axis; (2) the z -axis.
- (c) By revolving the line $x + z = 4$ about the z -axis.
- (d) By revolving the line $2x = 3y$ about the y -axis.
- (e) By revolving the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ about (1) the x -axis; (2) the y -axis.
- (f) By revolving the curve $y = \sin x$ about the x -axis.

2. Find the equation of the locus of points at a constant distance d from the point (x_1, y_1, z_1) . Can this surface be interpreted as a surface of revolution? What is this locus?

3. Find the equation of the locus of points equidistant from the yz -plane and the point $(a, 0, 0)$. Interpret the result.

4. Find the equation of the locus of points such that the sum of the distances from $(0, 0, 15)$ and $(0, 0, -15)$ is equal to 34. Interpret the result.

5. Find a generating curve for each of the following surfaces of revolution:

$$(a) x^2 + z^2 = 2y.$$

$$(b) x^2 + y^2 + z^2 = 16.$$

$$(c) \frac{x^2}{16} + \frac{y^2}{16} + \frac{z^2}{9} = 1.$$

$$(d) 4x^2 - y^2 + 4z^2 = 0.$$

$$(e) x^2 + y^2 - (z - 3)^2 = 0.$$

$$(f) z = e^{-(x^2 + y^2)}.$$

6. Find the equation of the conical surface of revolution which cuts the xz -plane in the curve $x^2 + z^2 = 16$ and whose vertex is at $(0, 3, 0)$.

7. Find the equation of the surface formed by revolving the circle $x^2 + (z - b)^2 = a^2$ ($a < b$) about the x -axis. (Such a surface is called a torus.)

136. Traces of a surface in the coördinate planes and in planes parallel to the coördinate planes. The curve in which a surface intersects a plane is called the *trace* of the surface in that plane. The trace of a surface in the xy -plane is found by intersecting the surface with the xy -plane, that is, by making $z = 0$ in the equation of the surface. Similarly, the xz - and yz -traces are found by making $y = 0$ and $x = 0$, respectively, in the equation of the surface. Thus the traces of the surface

$$4x^2 + y^2 - 2z = 8$$

in the xy -, yz -, and xz -planes, respectively, are

$$4x^2 + y^2 = 8, \quad y^2 - 2z = 8, \quad \text{and} \quad 4x^2 - 2z = 8.$$

The equation of any plane parallel to the xy -plane is of the form $z = k$. If we let $z = k$ in the equation of a surface, we have the equation of the trace of that surface in the plane $z = k$. Traces of a surface in planes parallel to the other coördinate planes are found in a similar manner.

137. Discussion of a surface. The discussion of a surface whose equation is given should include the following points:

(a) *Symmetry.* The same tests apply for symmetry with respect to the coördinate planes as those used in the plane for symmetry with respect to the axes.

(b) *Intercepts on the axes.*

(c) *Traces in the coördinate planes.*

(d) *Traces in planes parallel to each of the coördinate planes.*
If all traces in planes parallel to a coördinate plane are circles, the surface is a surface of revolution. Find the generating curve.

(e) *Extent of the surface.*

Very little can be learned concerning a surface by locating points on the surface.

138. Examples. **EXAMPLE 1.** Discuss the locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The surface is symmetric with respect to each of the coördinate planes. The intercepts are $\pm a$ on the x -axis, $\pm b$ on the y -axis, and $\pm c$ on the z -axis. The trace in the xy -plane is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This equation represents an ellipse with semiaxes a and b . Similarly, traces in the yz - and xz -planes are ellipses. Transposing z^2/c^2 , we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2}.$$

Letting $z = \pm k$ ($k > 0$) and dividing by $1 - k^2/c^2$, we obtain

$$\frac{x^2}{a^2\left(1 - \frac{k^2}{c^2}\right)} + \frac{y^2}{b^2\left(1 - \frac{k^2}{c^2}\right)} = 1.$$

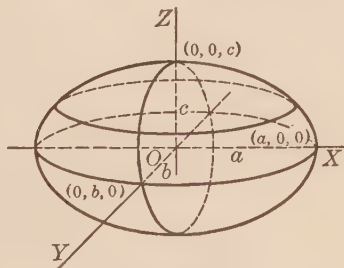


FIG. 123

The last equation shows that the sections of the surface by planes $z = \pm k$ are ellipses for $k < c$. The semiaxes $a\sqrt{1-k^2/c^2}$ and $b\sqrt{1-k^2/c^2}$ decrease as k approaches c in value. For $k = c$ the section is a point; and for $k > c$ the planes do not intersect the surface. Hence the surface lies between the planes $z = \pm c$, and each trace in a plane parallel to the xy -plane is an ellipse. Similarly, we can show that the surface lies between the planes $y = \pm b$ and between the planes $x = \pm a$, and that traces in planes parallel to the xz - and yz -planes, respectively, are ellipses. Such a surface is called an **ellipsoid**.

If $a = b$, traces in planes parallel to the xy -plane are circles and the surface is an ellipsoid of revolution. The generating curve is either the ellipse in the xz -plane,

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1,$$

or the ellipse in the yz -plane,

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Similarly, if $b = c$ or if $a = c$, the surface is an ellipsoid of revolution.

If $a = b = c$, traces in planes parallel to the coördinate planes are circles and the surface is a sphere whose equation is

$$x^2 + y^2 + z^2 = a^2.$$

EXAMPLE 2. Discuss the locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The surface is symmetric with respect to each of the coördinate planes. The intercepts are $\pm a$ on the x -axis and $\pm b$ on the y -axis. The surface does not cross the z -axis. The trace in the xy -plane is an ellipse and the traces in the yz - and the xz -planes are hyperbolas with transverse axes along the y - and the x -axes, respectively. Traces in planes $z = \pm k$ are

ellipses whose semiaxes, $a\sqrt{1+k^2/c^2}$ and $b\sqrt{1+k^2/c^2}$, increase as k increases. Traces in the planes $y = \pm h$ are hyperbolas,

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{h^2}{b^2},$$

with transverse axes in the xy -plane for values of h less than b . Traces in the planes $y = \pm h = \pm b$ are the straight lines

$$\frac{x}{a} \pm \frac{z}{c} = 0.$$

For values of h greater than b the traces are hyperbolas with transverse axes in the yz -plane.

Similarly, traces in planes parallel to the yz -plane are hyperbolas or straight lines. Such a surface is called a **hyperboloid of one sheet**.

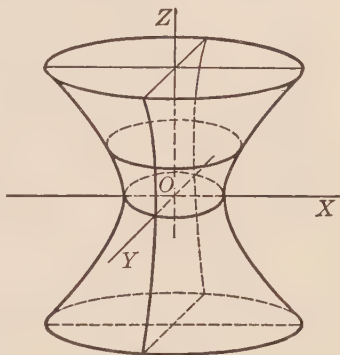


FIG. 124

EXAMPLE 3. Discuss the locus of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The surface is symmetric with respect to each of the coördinate planes. The intercepts are $\pm a$ on the x -axis.

The surface does not intersect the other axes. Traces in planes $x = \pm k$ parallel to the yz -plane are given by the equation

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1.$$

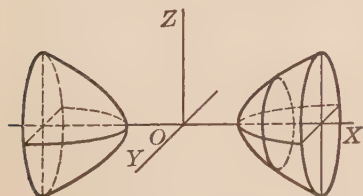


FIG. 125

This equation shows that for $k < a$ the planes do not intersect the surface and hence the surface is outside the two planes $x = \pm a$. For $k > a$ the sections are ellipses whose semiaxes increase as k increases. Traces in planes parallel to the xy - and

xz -planes, respectively, are hyperbolas. This surface is called an **elliptic hyperboloid of two sheets**. In case $b = c$, the surface is a surface of revolution.

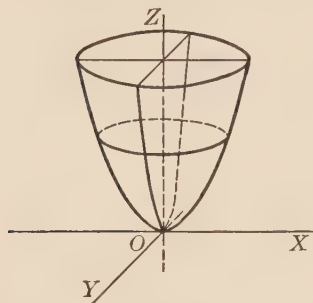


FIG. 126

EXAMPLE 4. Discuss the locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz.$$

The surface is symmetric with respect to the xz - and yz -planes. It touches the xy -plane at the origin but does not extend below that plane if c is positive, nor above that plane if c is negative.

The traces in the xz - and yz -planes are the parabolas $x^2 = a^2cz$ and $y^2 = b^2cz$, respectively. Traces in planes parallel to the xy -plane are ellipses with increasing semiaxes as the intersecting plane is moved farther and farther from the xy -plane. The surface is called an **elliptic paraboloid** if $a \neq b$, and a **paraboloid of revolution** if $a = b$. The figure is drawn for c positive.

EXAMPLE 5. Discuss the locus of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz.$$

The surface is symmetric with respect to the xz - and yz -planes. The origin is on the surface and the axes intersect the surface in no other point. The traces in the xy -, the yz -, and the xz -planes, respectively, are the lines $x/a \pm y/b = 0$, the parabola $y^2 = -b^2cz$, and the parabola $x^2 = a^2cz$. Traces of the surface in planes parallel to the xy -plane are hyperbolas with the transverse axes in the xz -plane when the intersecting planes are above the xy -plane and with the transverse axes in the yz -plane

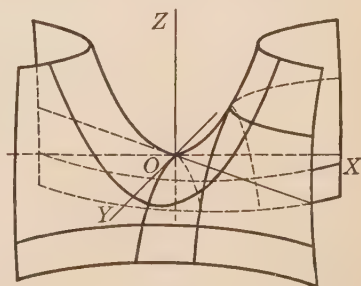


FIG. 127

when the intersecting planes are below the xy -plane. For c positive, as in Fig. 127, traces of the surface in planes parallel to the xz -plane are found to be parabolas, concave upward, with vertices on the parabola which is the trace of the surface in the yz -plane. Similarly, the traces in planes parallel to the yz -plane are found to be parabolas, concave downward, with vertices on the parabola which is the trace of the surface in the xz -plane. This surface is called a **hyperbolic paraboloid**.

EXAMPLE 6. Discuss the locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

The surface is symmetric with respect to each of the three coordinate planes. The origin is on the surface and there are no other intercepts on the axes. The trace in the xy -plane is a point; the trace in the xz -plane is the lines $x/a \pm z/c = 0$; and the trace in the yz -plane is the lines $y/b \pm z/c = 0$. Traces in planes parallel to the xy -plane are ellipses with increasing semiaxes as the intersecting plane is moved farther and farther from the xy -plane. Traces of the surface in planes parallel to the xz - and the yz -planes are hyperbolas with transverse axes parallel to the z -axis. The surface is an **elliptic cone** with axis along the z -axis. In case $a = b$, the surface is a right circular cone.

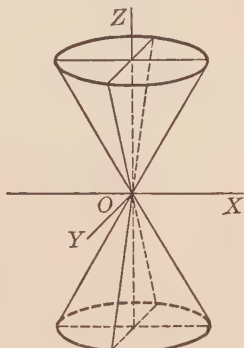


FIG. 128

EXERCISES

Discuss the surfaces represented by the following equations:

1. $\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1.$

4. $\frac{x^2}{25} - \frac{y^2}{16} + \frac{z^2}{9} = 0.$

2. $\frac{x^2}{25} - \frac{y^2}{16} + \frac{z^2}{9} = 1.$

5. $\frac{x^2}{25} - \frac{y^2}{16} + \frac{z^2}{9} = -1.$

3. $\frac{x^2}{25} - \frac{y^2}{16} - \frac{z^2}{9} = 1.$

6. $\frac{x^2}{25} - \frac{y^2}{16} - \frac{z^2}{9} = 0.$

Discuss the surfaces represented by the following equations:

7. $x^2 + y^2 + z^2 = 20$.

11. $x^2 + y^2 + z^2 - 2ax = 0$.

8. $9x^2 + 4y^2 + z^2 = 36$.

12. $(x-2)^2 = 4y^2 + z^2$.

~~9.~~ $y^2 + 2z^2 = 8x$.

13. $\frac{x^2}{9} - \frac{y^2}{16} = 2z$.

10. $4x^2 - 9y^2 = 0$.

14. $4z = 16 - x^2 - y^2$.

15. Show that $x^2 - y^2 = az$ becomes $2x'y' = az$ by rotating the x - and y -axes through an angle of -45° , keeping the z -axis fixed.

Discuss the surfaces represented by the following equations:

16. $2z = 1 - x - y^2$.

17. $(x+y)^2 + z^2 = 9$.

139. The locus of an equation in three variables. The elimination of z between the equations $f(x, y, z) = 0$ and $z = c$ results in the equation

$$f(x, y, c) = 0.$$

This equation represents a cylindrical surface with elements parallel to the z -axis. This surface cuts the xy -plane in a curve which is congruent to the curve cut from the plane $z = c$ by the same surface. Hence we may look upon $f(x, y, c) = 0$ as the equation of a cylindrical surface or as the equation of the curve in which that cylindrical surface cuts the xy -plane. But the equation $f(x, y, c) = 0$ is satisfied by all the points whose coördinates satisfy the equations $f(x, y, z) = 0$ and $z = c$ simultaneously. Hence the plane $z = c$ intersects the locus represented by $f(x, y, z) = 0$ in a plane curve which is congruent to the curve in the xy -plane represented by $f(x, y, c) = 0$. It was pointed out in Sec. 42 that an equation in two variables, such as $f(x, y, c) = 0$, represents in a plane either a curve, isolated points, or no real locus. Hence the plane $z = c$ may intersect the locus of $f(x, y, z) = 0$ in a plane curve, in isolated points, or in no real locus. A similar statement holds for planes parallel to the other coördinate planes. Hence the locus represented by an equation in three variables, $f(x, y, z) = 0$, is a surface, isolated points or lines, or no real locus.

140. Intersection of surfaces. Curves. Two surfaces will, in general, intersect in a curve. The equations of two sur-

faces, therefore, considered simultaneously represent the curve of intersection of the two surfaces. From the equations of two surfaces, $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$, (3)

we obtain by eliminating one of the variables, say z , one equation in two variables,

$$\phi(x, y) = 0. \quad (4)$$

Equation (4) represents a cylindrical surface with elements parallel to the z -axis and is satisfied by all the points whose coördinates satisfy equations (3) simultaneously. Equation (4) represents, therefore, a cylindrical surface through the curve of intersection of the surfaces represented by equations (3). This surface intersects the xy -plane in a curve whose equation is

$$\phi(x, y) = 0.$$

This curve is the projection upon the xy -plane of the curve of intersection of the surfaces represented by equations (3). Similarly, the elimination of y and of x , respectively, from equations (3) will give the cylindrical surfaces which project the curve of intersection of the surfaces upon the xz - and the yz -planes.

In general, the curve of intersection of two surfaces is not a plane curve. A curve which does not lie in a plane is called a **skew curve**.

EXAMPLE 1. Find the equations of the projections upon the coördinate planes of the curve of intersection of the ellipsoid $x^2 + 2y^2 + 9z^2 = 36$ and the plane $y = 2x$.

Eliminating x , y , and z in turn, we obtain

$$y^2 + 4z^2 = 16, \quad x^2 + z^2 = 4, \quad \text{and} \quad y = 2x$$

as the equations of the projections of the curve of intersection of the two surfaces upon the yz -, the xz -, and the xy -planes, respectively.

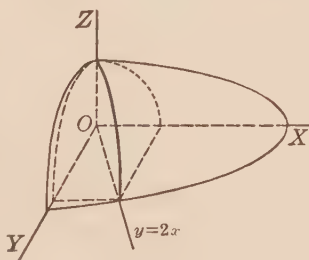


FIG. 129

EXERCISES

Find the equations of the projections upon the coördinate planes of the curves of intersection of the surfaces whose equations are given, and make a sketch in each case:

1. $x^2 + y^2 + z^2 = 16$ and $y = \sqrt{3} x$.
2. $y = x^2$ and $x^2 + z^2 = 1$.
3. $x^2 + y^2 = 2 ax$ and $x^2 + y^2 = az$.
4. $y^2 + z^2 = 2 x$ and $x - y = 4$.
5. $x^2 = y^2 + 3 z^2$ and $2 x + y = 4$.
6. $x^2 = y^2 + 3 z^2$ and $x + y = 3$.
7. $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 - 4 x = 0$.
8. $x^2 + y^2 + z^2 = 16$ and $z = x + y$.
9. $z = 1 - x^2$ and $x = 1 - y^2$.
10. $2 z = 4 - x^2 - y^2$ and $y + z = 2$.

11. Picture the volume in the first octant bounded by the surface $x = 4 - z - y^2$ and lying under the plane $y + z = 2$.

12. Picture the volume bounded by the surfaces $y^2 + z^2 = 4$ and $x^2 + z^2 = 4$.

13. A right circular cylinder with altitude equal to the diameter of the base is cut by a plane which cuts one base in a diameter and is tangent to the other base. Picture the smaller of the two parts into which the cylinder is divided. Select axes and write the equations of the surfaces which bound the part pictured.

141. Spherical coördinates. Another method of locating a point in space is by giving its distance and direction from a fixed point. In Fig. 130 let O be the fixed point and draw OX and OZ making $\angle XOZ = 90^\circ$. Let θ be the angle between the plane POZ and the plane XOZ and let ϕ be the angle between the lines OP and OZ . The point P is determined by ρ , θ , and ϕ .

If OX and OZ are chosen for the x - and z -axes, respectively, and OY is drawn perpendicular to OX and OZ , relations between the spherical and rectangular coördinates of P are easily found. Draw PM perpendicular to the plane XOY and join O and M . Then $\angle MOP = 90^\circ - \phi$, and

$OM = OP \cos \angle MOP = \rho \cos (90^\circ - \phi) = \rho \sin \phi$. The rectangular coördinates are given in terms of the spherical coördinates by the equations

$$x = OA = OM \cos \theta = \rho \sin \phi \cos \theta,$$

$$y = AM = OM \sin \theta = \rho \sin \phi \sin \theta,$$

$$z = MP = \rho \cos \phi.$$

The spherical coördinates can be expressed in terms of the rectangular coördinates by solving the foregoing equations for ρ, ϕ , and θ . Squaring and adding, we have

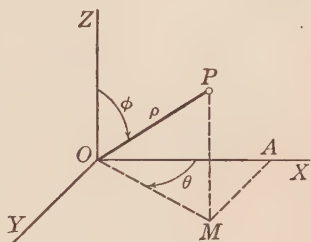


FIG. 130

$$x^2 + y^2 + z^2 = \rho^2, \quad \text{or} \quad \rho = \pm \sqrt{x^2 + y^2 + z^2}.$$

Then from the third of the foregoing equations we obtain

$$\phi = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}},$$

and from the first and second equations we obtain

$$\theta = \tan^{-1} \frac{y}{x}.$$

142. Cylindrical coördinates. Cylindrical coördinates are a combination of polar and rectangular coördinates. Thus in Fig. 131 the cylindrical coördinates of P are (r, θ, z) , where r and θ are the polar coördinates of M , the projection of P upon the xy -plane, and where z is the distance MP .

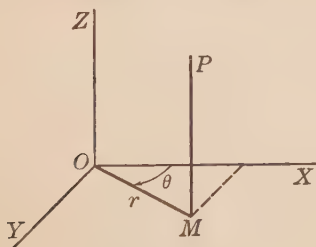


FIG. 131

The rectangular coördinates (x, y, z) of P are related to the cylindrical coördinates (r, θ, z) of P by the equations

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

The cylindrical coördinates (r, θ, z) of P are related to the spherical coördinates (ρ, θ, ϕ) of P by the equations

$$r = \rho \sin \phi, \quad \theta = \theta, \quad z = \rho \cos \phi.$$

EXERCISES

1. Write the rectangular and the cylindrical coördinates of the points whose spherical coördinates are

$$(a) \left(8, \frac{\pi}{4}, \frac{\pi}{3}\right).$$

$$(c) \left(2, 0, \frac{\pi}{4}\right).$$

$$(b) \left(8, \frac{\pi}{4}, \frac{2\pi}{3}\right).$$

$$(d) \left(3, \frac{\pi}{2}, \frac{\pi}{2}\right).$$

2. Write the rectangular and the spherical coördinates of the points whose cylindrical coördinates are

$$(a) \left(5, \frac{\pi}{6}, 6\right).$$

$$(c) \left(4, \frac{\pi}{3}, -2\right).$$

$$(b) \left(-5, -\frac{\pi}{6}, 6\right).$$

$$(d) (4, 0, 0).$$

3. Write the spherical and the cylindrical coördinates of the points whose rectangular coördinates are

$$(a) (6, 3, 0).$$

$$(c) (0, 4, 2).$$

$$(b) (-5, 0, 0).$$

$$(d) (4\sqrt{3}, 4, 8).$$

4. Determine the loci whose equations in spherical coördinates are

$$(a) \rho = a.$$

$$(f) \rho = 4, \phi = \frac{\pi}{4}.$$

$$(b) \theta = \frac{\pi}{4}.$$

$$(g) \rho = 4, \phi = \frac{\pi}{4}, \theta = \frac{\pi}{3}.$$

$$(c) \phi = \frac{\pi}{6}.$$

$$(h) \rho \cos \phi = 6, \phi = \frac{\pi}{6}.$$

$$(d) \phi = \frac{\pi}{4}, \theta = \frac{\pi}{6}.$$

$$(i) \rho \cos \phi = 6, \theta = \frac{\pi}{4}.$$

$$(e) \rho = 4, \theta = \frac{\pi}{3}.$$

$$(j) \rho \cos \theta = 4, \phi = \frac{\pi}{2}.$$

5. Determine the loci whose equations in cylindrical coördinates are

$$(a) r = a.$$

$$(e) \theta = \frac{\pi}{6}, z = 6.$$

$$(b) \theta = \frac{\pi}{3}.$$

$$(f) r \sin \theta = 3, z = 5.$$

$$(c) z = k.$$

$$(g) r \sin \theta = 3, r = 6.$$

$$(d) r = a, z = 6.$$

$$(h) r = 0.$$

6. Transform the following equations into equations in cylindrical or spherical coordinates. Choose the form which gives the simpler equation.

$$(a) x^2 + y^2 = 5z.$$

$$(c) x^2 + z^2 = 3y.$$

$$(b) xz = ay.$$

$$(d) (x^2 + y^2 + z^2)^2 = 3axyz.$$

7. Transform the following equations into equations in rectangular coordinates:

$$(a) \rho = 3 \sin 2\theta \cos \phi.$$

$$(c) z = a - r.$$

$$(b) \rho^2 = 2 \cos 2\theta.$$

$$(d) r^2 + a^2 z^2 = b^2.$$

8. Discuss the equations

$$(a) \rho = 8 \cos \phi.$$

$$(d) a^2 r^2 - b^2 z^2 = c^2.$$

$$(b) \rho - 6 \sin \phi \cos \theta = 0.$$

$$(e) b^2 z^2 - a^2 r^2 = c^2.$$

$$(c) r = 2a \sin \theta.$$

$$(f) \rho = a \csc 2\theta \cot \phi \csc \phi.$$

9. A hole 2 inches in diameter is bored through the center of a sphere 12 inches in diameter. Choose axes and write the equations of the bounding surfaces of the part removed, (a) in rectangular coordinates; (b) in spherical coordinates; (c) in cylindrical coordinates.

10. A square hole with side 2 inches is cut through a cylindrical post 6 inches in diameter so that the axis of the hole intersects the axis of the cylinder at right angles. Choose axes and write the equations of the bounding surfaces of the part removed, (a) in rectangular coordinates; (b) in spherical coordinates; (c) in cylindrical coordinates.

MISCELLANEOUS EXERCISES

1. What surfaces are represented by the following equations?

$$(a) x^2 + z^2 = a^2.$$

$$(f) x^2 + y^2 = z^2.$$

$$(b) x^2 = 2py.$$

$$(g) x^2 + 2y^2 + 3z^2 = 4.$$

$$(c) y = mx + b.$$

$$(h) xy = z^2.$$

$$(d) y^2 - z^2 = 0.$$

$$(i) x + y = z^2.$$

$$(e) xyz = 0.$$

$$(j) x^2 + y^2 = z.$$

2. Find the equation of the right circular cylinder whose axis is the line $x = 3$, $y = 4$ and whose radius is 5.

3. Find the equation of the cone with vertex at the origin and axis along the x -axis if the cross-section made by the plane $x = 4$ is an ellipse whose axes are parallel to the y - and z -axes and are 2 units and 6 units, respectively.

4. Find the equation of the sphere which passes through $(0, 0, 0)$, $(1, 3, 2)$, $(1, 2, -3)$, and $(5, 3, 6)$.

5. Show that the plane $5x - y - 4z + 31 = 0$ is tangent to the sphere $x^2 + y^2 + z^2 - 8x - 2y - 4z = 21$, and write the equation of the parallel tangent plane.

6. Find the points where the line $x - 2y - 2z + 6 = 0$, $x + 2z - 2 = 0$ intersects the cone $4x^2 + 4z^2 - y^2 = 0$.

7. Find the equation of the surface of revolution generated by revolving the hyperbola $xy = a^2$ about one of its asymptotes.

8. The arc of the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ from $(a, 0)$ to $(0, a)$ in the xy -plane is revolved about the x -axis. Find the equation of the surface thus generated.

9. The four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is revolved about the x -axis. Find the equation of the surface thus generated.

10. The line $x + z = 2$ is revolved about the line $z = 2$ in the xz -plane. Find the equation of the surface thus generated.

11. Find the equation of the surface of revolution generated by revolving the arc of the parabola $y^2 = 4px$ terminated by the ends of the latus rectum about the latus rectum.

12. Discuss the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$.

13. A circular cylinder of diameter 6 inches intersects a sphere of radius 6 inches so that an element of the cylinder coincides with a diameter of the sphere. Choose axes and write the equations of the bounding surfaces of the common volume in (a) rectangular coordinates; (b) spherical coordinates; (c) cylindrical coordinates.

14. Show that in general the parametric equations $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$ represent a skew curve.

15. Discuss the curve $x = t$, $y = t^2$, $z = t^3$.

16. Discuss the curve $x = a \cos t$, $y = a \sin t$, $z = kt$. (This curve is called a helix.)

ANSWERS

Page 6

5. (a) 12. (b) -15.
6. (a) 13. (c) 52.

7. (a) 65.5. (b) 140.
8. 45.5.

Pages 8-9

1. (a) 10. (b) $4\sqrt{13}$.
2. (a) $2\sqrt{b^2 + d^2}$.
3. 36.
7. (a) (2, 4).
8. (2, -8).
9. $3\sqrt{13}$, 12, $3\sqrt{13}$.
10. (1, 1).
11. (7.2, 4.8).
12. (a) (14, 26). (b) (17, 31).
13. (4, 4), (7, 10).
14. $(1 + 3\sqrt{10}, 3 + \sqrt{10})$.
15. $(0, \frac{5}{3})$.
16. (6, 6).
17. (1, -2).
18. (-2, 1), (10, -3).
19. (7, 1), (1, -7).

$$20. \left(\frac{3 \pm \sqrt{3}}{2}, \frac{-1 \mp 7\sqrt{3}}{2} \right).$$

Pages 12-13

1. (a) 3. (b) $\frac{4}{3}$.
9. $81^\circ 52.2'$.
10. $37^\circ 11'$.
12. $26^\circ 33.9'$, $78^\circ 41.4'$, $74^\circ 44.7'$.
14. (1, 9), (-3, 3), (7, -1).
15. $-5, \frac{1}{5}$.
17. $1, -2 + \sqrt{3}, -2 - \sqrt{3}$.

Pages 14-15

4. $(-1, \frac{4\pi}{3})$, $(1, -\frac{5\pi}{3})$, $(-1, -\frac{2\pi}{3})$.
9. $2\sqrt{13}$.
11. 30.
12. (a) $18\sqrt{3}$. (b) $\frac{35\sqrt{3}}{2}$.

Pages 18-19

4. $x - 2y = 6$.
5. $x - y + 4 = 0$.
6. $4x - 3y + 7 = 0$.
7. $3x + 4y = 26$.
12. $5x - 2y = 4$.

Pages 21-23

4. (a) $2x - y = 9$. (b) $2x + 3y + 3 = 0$. (e) $bx - ay = 0$. (f) $x - y = 2\sqrt{2}$.
 5. (a) $2x - y + 1 = 0$. (d) $8x + 3y = 6$.
 (b) $3x + 4y = 24$. (j) $dx - by = ad - bc$.
 8. $x - 2y = 0$, $4x + 7y = 40$, $8x - y = 40$.
 10. $y = \sqrt{3}x$, $\sqrt{3}x + y = \sqrt{3}a$, $y = 0$.
 12. $8x + 3y = 0$. 14. $5x - 3y = 4$, $3x + 5y = 16$.
 13. $3x + 4y = 1$, $4x - 3y = 18$. 15. $2x + 18y = 41$, $18x + 2y = 49$.

Pages 25-27

1. (a) $y = -\frac{1}{3}x + \frac{1}{3}$, $m = -\frac{1}{3}$, $b = \frac{1}{3}$. 9. $6x + y = 0$, $2x + 5y = 28$,
 $x - y = 7$.
 4. $5x + 3y = 0$, $3x - 5y + 34 = 0$. 10. $3\sqrt{10}$, $6\sqrt{2}$.
 5. $2x - 3y = 13$. 11. Vertices are $(-5, -4)$, $(3, 0)$.
 6. $(1, -\frac{2}{3})$. 12. $63^\circ 26.1'$.
 7. $(-1, -7)$. 13. $56^\circ 18.6'$.
 8. $2\sqrt{10}$. 14. One angle is $53^\circ 7.8'$.
 15. One angle is $36^\circ 1.7'$.

Pages 31-33

2. (a) $\frac{3x}{5} - \frac{4y}{5} - 2 = 0$, $\omega = \tan^{-1}\frac{4}{3}$, $p = 2$.
 (d) $\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = 0$, $\omega = \frac{\pi}{4}$, $p = 0$.
 4. (a) 4. (d) $-\frac{2}{3}$. 13. 88.
 7. (a) $\sqrt{5}$. 14. $8x + 8y - 15 = 0$,
 $4x - 4y + 9 = 0$.
 8. One altitude is $4\sqrt{5}$. 15. $3x + y = 0$, $x - 3y = 0$.
 9. $12x - 5y = 78$, $12x - 5y = 0$. 16. $7x + y = 3$, $12x - 14y = 13$,
 $x + 13y + 6 = 0$.
 10. (a) $4x + 2y = 25$. 17. 2.5.
 11. $(3, 6)$, $(-1, -2)$.
 12. 15.

Pages 36-37

2. (a) $y = 2x + 3$. 4. $5x - 3y = k$, $5x - 3y = 12$.
 3. (a) $2x + y = 2$. 5. $x + 2y = k$, $x + 2y = 12$.
 6. $x \cos \omega + y \sin \omega - 5 = 0$, $4x + 3y - 25 = 0$.
 7. $3x - 4y \pm 25 = 0$.
 8. (a) $x + 2y = 10$, $8x + y = 20$. (b) $2x + y = 8$. (c) Impossible.
 10. (a) $4x - 13y + 21 = 0$. (b) $5x + 3y = 0$.
 11. (a) $x - 2y + 3 = 0$. (b) $2x - y = 0$.
 12. (a) $x - y - 2 = 0$.
 13. $2x + y - 5 = 0$.

Page 38

5. (a) $x = 3$. (c) $y = \sqrt{3}x$. (e) $x + y = 4$.
 6. (a) $\rho = 2 \sec \theta$. (c) $\theta = \tan^{-1}(-\frac{1}{2})$. (e) $\rho = \frac{\sqrt{2}}{2} \sec\left(\theta - \frac{\pi}{4}\right)$.
 (f) $\rho = \frac{-C}{\sqrt{A^2 + B^2}} \sec\left(\theta - \tan^{-1} \frac{B}{A}\right)$.
 10. $\rho = \pm \frac{a\sqrt{3}}{2} \csc \theta$, $\rho = \pm \frac{a\sqrt{3}}{2} \sec\left(\theta + \frac{\pi}{6}\right)$, $\rho = \pm \frac{a\sqrt{3}}{2} \sec\left(\theta - \frac{\pi}{6}\right)$.

Pages 39-42. Miscellaneous Exercises

3. $(-4, 6)$.
 4. $(4, 1)$.
 5. -9 .
 6. $3x - 10y = 0$, $6x - 5y = 0$.
 7. $(-\frac{5}{8}, \frac{1}{2})$.
 8. (a) $10, \frac{5\sqrt{10}}{2}, \frac{5\sqrt{2}}{2}$.
 (b) One angle $116^\circ 34'$. (c) 12.5.
 9. $x + y = a$, $y = (2 - \sqrt{3})x + a$,
 $y = (2 + \sqrt{3})(x - a)$.
 12. $x - 7y = 40$,
 $x - 7y + 20 = 0$.
 13. 75, 25.
 14. 29.5.
 16. $y = 4x$, $x + 4y = 0$.
 17. $3x - 5y + 9 = 0$,
 $5x + 3y + 15 = 0$,
 $3x - 5y + 60 = 0$,
 $5x + 3y - 36 = 0$.
 18. $(4, 2)$, $(0, -3)$,
 $(-8, -6)$, $(-4, -1)$.
 19. $8x - y = 6$, $16x - 2y = 61$.
 20. 40.
 21. $4\sqrt{10}$.
 22. $2x + 2y = 15$.
 23. $3x - 5y = 30$.
 24. $3x + 7y = 21$,
 $x - 2y = 20$.
 25. $3x + 16y = 60$, $3x + y = 15$.
 26. $3x + 4y - 30 = 0$, $y = 6$.
 27. $x + 2y = 20$, $2x + y = 20$.
 28. $7x + 24y = 21$, $4x - 3y = 12$.
 31. $x - 2y = 3$.
 32. 220 feet.

Pages 55-56

16. $x^3 = 2y^2$.
 17. $4y = x^3$.
 18. $xy = 1$.
 19. $x^2 + y^2 = a^2$.
 20. $x^2 + y^2 = 10x$.
 21. $y^2 = 2x + 1$.
 22. $xy = x + y$.
 23. $\rho = 10 \cos \theta$.
 24. $\rho = -10 \sin \theta$.
 25. $\rho = 10 \cos \theta \sec 2\theta$.
 26. $\rho^2 = \cos 2\theta \sec^4 \theta$.

Pages 57-58

1. $(9, 9)$, $(1, -3)$.
 2. $(1, 3)$.
 3. Do not intersect.
 4. $(4, -3)$, $(-4, 3)$.
 5. $(0, 0)$, $(2, 3)$.
 6. $(0, 0)$, $(4, 8)$, $(-4, -8)$.
 7. Do not intersect.
 8. $(4, 8)$, $(1, 1)$,
 $(\frac{4}{3}, -\frac{8}{3})$.
 9. $(4, 2)$, $(2, 4)$, $(-4, -2)$,
 $(-2, -4)$.
 10. $(3, 0)$.
 11. $(2, \pm \sqrt{5})$, $(-3, 0)$.
 12. $(-1, \pm \sqrt{3})$.

15. $(2, 2)$, $(-1 + \sqrt{3}, -1 - \sqrt{3})$, $(-1 - \sqrt{3}, -1 + \sqrt{3})$.
 16. $(2, \frac{\pi}{4})$, $(2, -\frac{3\pi}{4})$, $(-2, -\frac{\pi}{4})$, $(-2, \frac{3\pi}{4})$.
 17. $(6, \pm \frac{\pi}{3})$. 20. $(3, \pm \frac{\pi}{3})$, $(3, \pm \frac{2\pi}{3})$.

Pages 60-61

1. $x^2 + y^2 = 100$. 9. $9x^2 - 16y^2 = 144$.
 2. $x^2 + y^2 - 6x - 8y = 0$. 10. $3x^2 + 4y^2 - 12x = 0$.
 4. $y^2(1 - x^2) = x^4$. 11. $x^2 - 3y^2 - 12y = 0$.
 5. $4x - 3y = 16$. 12. $3x^2 - 2xy + 3y^2 - 8 = 0$.
 6. $y^2 = 8x - 16$. 13. $x^2 + 2xy + y^2 + 8\sqrt{2}x - 8\sqrt{2}y = 0$.
 7. $x^2 - 4x - 4y = 0$. 14. $xy - 3x + 2y = 0$.
 8. $16x^2 + 25y^2 = 400$.

Pages 64-65

1. $x'^2 + y'^2 = 25$. 4. $(-5, \frac{3}{2})$. 10. $x'^2 - y'^2 = a^2$.
 2. $y'^2 = 2x'$. 9. $2x'y' = a^2$. 11. $x'^2 + 2y'^2 = 4$.
 12. $y'^2 + 8x' = 0$. 13. $4x'^2 + 25y'^2 = 100$.

Pages 67-69. Miscellaneous Exercises

2. (a) $(0, 0)$, $(3, 1)$, $(-3, -1)$. 4. (a) $5\sqrt{2}$, 7. (b) $89^\circ 11'$.
 (b) $(1, 1)$, $(-3, -1)$. 6. $A + C = 0$.
 3. $x + y = 1$. 8. $\rho = \tan \theta$.
 9. $x^2 + y^2 - ax - ay = 0$.
 18. (a) -5 . (b) $\pm 2\sqrt{3}$. (c) ± 5 . (d) 1.2619. (e) -0.6309 . (f) $\frac{1}{27}$.

Pages 71-73

1. (a) $x^2 + y^2 = 9$. (b) $x^2 + y^2 = 20$. (c) $x^2 + y^2 - 4x - 8y = 0$.
 2. (a) $x^2 + y^2 - 2rx - 2ry + r^2 = 0$. (b) $x^2 + y^2 + 2rx - 2ry + r^2 = 0$.
 3. $x^2 + y^2 - 5x + 12y = 0$.
 4. (a) Center $(1, 3)$, $r = 4$. (d) Center $(6, 0)$, $r = 6$.
 5. $3x - 5y = 9$.
 9. $x^2 + y^2 + 5x - 6y - 28 = 0$.

Pages 75-76

1. (a) $x^2 + y^2 - 5x - y + 4 = 0$. 5. $x^2 + y^2 - 6x - 12y + 5 = 0$.
 2. $4x^2 + 4y^2 - 50x - 35y + 115 = 0$. 6. $x^2 + y^2 + 12x - 12y - 106 = 0$.
 3. $x^2 + y^2 + 6x - 8y - 75 = 0$. 7. $x^2 + y^2 - 16x - 14y + 49 = 0$.
 4. $x^2 + y^2 - 6y - 16 = 0$. 8. $x^2 + y^2 + 10x - 6y + 9 = 0$.

9. $x^2 + y^2 + 20x + 68 = 0$, $x^2 + y^2 - 12x + 4 = 0$.
 10. $x^2 + y^2 + 6x - 8y + 5 = 0$.
 11. $x^2 + y^2 - 12x - 18y + 17 = 0$, $x^2 + y^2 + 4x + 14y - 47 = 0$.
 12. $x^2 + y^2 - 20x - 20y + 100 = 0$, $x^2 + y^2 - 4x - 4y + 4 = 0$.
 13. $x^2 + y^2 - 20y = 0$, $x^2 + y^2 - 20x - 10y + 100 = 0$.
 14. $x^2 + y^2 - 8x - 2y + 7 = 0$, $x^2 + y^2 - 3x - 7y + 12 = 0$.
 15. $2x^2 + 2y^2 - 4x - 20y - 29 = 0$.

Pages 78-79

1. (a) $x^2 + y^2 - 20x + 8y + 31 = 0$. (b) $4x^2 + 4y^2 - 18x + y = 0$.
 2. $x^2 + y^2 - 2x + 2y - 16 = 0$.
 3. (a) $6x^2 + 6y^2 - 34y + 17 = 0$. (b) $5x^2 + 5y^2 - 17x - 17y = 0$.
 5. $5\sqrt{2}$. 6. $4\sqrt{5}$.

Pages 80-81

8. (a) $\rho = 4 \cos \theta$. (d) $\rho = a(\sin \theta + \cos \theta)$.
 (b) $\rho = -5 \sin \theta$. (e) $\rho = 4(\cos \theta + \sin \theta \pm \sqrt{\sin 2\theta})$.
 (c) $\rho = a$. (f) $\rho = b \sin \theta - a \cos \theta$.

Pages 81-83. Miscellaneous Exercises

3. (a) $c = 8$. (b) $c = \frac{25}{8}$. 8. $x^2 + y^2 - 2x - 6y = 0$.
 4. $x^2 + y^2 = 9$. 9. $3\sqrt{5}$.
 5. $x^2 + y^2 = 4$, $x^2 + y^2 = 100$. 10. (2, 3).
 6. 10. 11. $3x^2 + 3y^2 - 2a\sqrt{3}y - 3a^2 = 0$.
 7. $x^2 + y^2 - 4x + y - 2 = 0$. 13. $4x^2 + 4y^2 - 20x - 20y + 25 = 0$.
 14. $x^2 + y^2 - 4x - 16y + 43 = 0$, $5x^2 + 5y^2 - 12x - 24y + 31 = 0$.
 15. $x^2 + y^2 - 14x - 10y + 34 = 0$, $x^2 + y^2 + 10x - 2y - 14 = 0$.
 16. $x^2 + y^2 - 2x - 8y + 9 = 0$. 20. $x + 2y = 10$.
 17. $x^2 + y^2 - 4x - 12y + 30 = 0$. 21. $2x - 3y = 18$.
 18. (2, 3), (6, 5). 22. $(8, \frac{3}{2})$, $53^\circ 8'$.
 19. (-1, 5), (7, 1). 23. (16, 0), (-4, 0).
 25. (2, 1).
 27. $\rho = 8 \cos \theta$, $\rho = 6 \sin \theta$, $\rho = 8 \cos \theta + 6 \sin \theta$.

Pages 86-87

1. (a) Focus (1, 0), directrix $x = -1$. 5. (a) $y^2 + 8x - 16 = 0$.
 (b) Focus $(-\frac{3}{2}, 0)$, directrix $x = \frac{3}{2}$. (c) $x^2 - 2x + 8y + 9 = 0$.
 (c) Focus (0, 3), directrix $y = -3$. 6. $2p$.
 2. (a) $3y^2 = x$, $x^2 = 9y$. 7. $x^2 + y^2 - 6x - 27 = 0$.
 (b) $5y^2 = 9x$, $3x^2 = -25y$. 9. 9 feet.
 4. $y^2 = 2px - p^2$.

Pages 90-91

1. (a) $y^2 - 6y - 12x + 21 = 0$. (c) $x^2 - 6x - 2y + 10 = 0$.
(e) $y^2 - 2y - 10x - 24 = 0$.
2. (a) Vertex (3, 1), focus (5, 1), latus rectum 8, directrix $x = 1$.
(c) Vertex (2, -6), focus (2, $-\frac{11}{2}$), latus rectum 2, directrix $y = -\frac{13}{2}$.
3. $10\sqrt{2}$. 8. (a) $2x^2 - 3x - y - 1 = 0$.
4. (a) $y^2 - 2y - 3x + 7 = 0$. (b) $2y^2 - y + 9x - 19 = 0$.
- (b) $x^2 - 4x + 3y + 1 = 0$. 10. $4x^2 - 4xy + y^2 - 8x - 16y + 24 = 0$.

Page 93

1. (a) Semiaxes 6, 4, foci ($\pm 2\sqrt{5}$, 0). 4. $x^2 + 3y^2 = 28$.
(b) Semiaxes 5, 3, foci (0, ± 4). 5. $x^2 + 4y^2 = 36$.
(c) Semiaxes $2\sqrt{3}$, 2, foci
($\pm 2\sqrt{2}$, 0). 6. $\frac{2b^2}{a}$.
2. (a) $3x^2 + 4y^2 = 48$. 7. $5y^2 = x$.
(b) $34x^2 + 25y^2 = 850$. 8. $25x^2 + 9y^2 = 900$.
3. (a) $3x^2 + 4y^2 = 3a^2$. 9. $3x^2 + 4y^2 - 18x - 8y - 17 = 0$.
(b) $x^2 + 2y^2 = a^2$. 10. (a) 2.01 feet. (b) 5.08 feet.

Pages 98-99

1. (a) $9(x-4)^2 + 25(y-2)^2 = 225$.
(b) $9x^2 + 8y^2 + 18x - 48y - 207 = 0$.
2. (a) Center (-2, 4), semiaxes $6\sqrt{2}$, 6, foci (-8, 4), (4, 4).
(b) Center (-2, -3), semiaxes $2\sqrt{6}$, $2\sqrt{2}$, foci (-2, -7), (-2, 1).
3. (a) $e = \frac{3}{5}$, directrices $3x = \pm 25$.
(c) $e = \frac{2}{3}$, directrices $2x = 15$, $2x + 3 = 0$.
4. (a) $3x^2 + 4y^2 = 108$. (d) $x^2 + 2y^2 - 4x - 12y - 10 = 0$.
6. $x^2 + 4y^2 - 4x - 8y - 92 = 0$.
8. $7x^2 - 2xy + 7y^2 - 24x - 24y = 0$.

Pages 101-102

1. (a) Semiaxes 4, 3, foci (± 5 , 0), asymptotes $3x \pm 4y = 0$.
(b) Semiaxes $2\sqrt{5}$, 4, foci (0, ± 6), asymptotes $\sqrt{5}x \pm 2y = 0$.
2. (a) $5x^2 - 4y^2 = 80$. 6. $x^2 - 3y^2 = 6$.
(b) $9x^2 - 40y^2 + 360 = 0$. 7. $\frac{2b^2}{a}$.
3. $3x^2 - y^2 = 3a^2$. 8. $x^2 + 12y^2 - 16x = 0$.
4. $16x^2 - 9y^2 = 576$. 9. $x^2 - 8y^2 + 32 = 0$.
5. $3x^2 - y^2 = 12$. 10. $8x^2 - y^2 - 32x + 6y + 15 = 0$.

Pages 106-107

1. (a) Center $(2, 3)$, semiaxes $3\sqrt{3}, 3$, foci $(-4, 3), (8, 3)$, asymptotes $x + \sqrt{3}y = 2 + 3\sqrt{3}, x - \sqrt{3}y = 2 - 3\sqrt{3}$.
 (b) Center $(-3, 4)$, semiaxes $4, 3$, foci $(-3, -1), (-3, 9)$, asymptotes $4x + 3y = 0, 4x - 3y + 24 = 0$.
2. (a) $e = \frac{4}{3}$, directrices $4x = \pm 9$. (b) $e = 3$, directrices $3y = \pm 4$.
3. (a) $5x^2 - 4y^2 = 20$. (d) $3x^2 - y^2 - 24x = 0$.
5. Vertices $(2\sqrt{2}, 2\sqrt{2}), (-2\sqrt{2}, -2\sqrt{2})$, axes $8, 8$, foci $(4, 4), (-4, -4)$, directrices $x + y = \pm 4$.
9. $3x^2 - 4y^2 - 24x - 16y + 20 = 0$.
10. $x^2 - 4xy + y^2 + 12x - 12y = 0$.

Pages 110-111

1. (a) Parabola $y'^2 = x'$. (a) Ellipse.
- (c) Ellipse $x'^2 + 2y'^2 - x' = 0$. 4. $2x'^2 + y'^2 + 4x' + y' + 3 = 0$.
- (e) Hyperbola $x'^2 - y'^2 = 9$. 11. $x^2 - 2xy + y^2 - 3x + 5y = 0$.

Pages 113-116. Miscellaneous Exercises

4. $3\sqrt{5}$. 13. $\frac{7}{8}$.
5. $(4 + \frac{4}{5}\sqrt{5}, \pm \frac{4}{5}\sqrt{5}), (4 - \frac{4}{5}\sqrt{5}, \pm \frac{4}{5}\sqrt{5})$. 15. 32.5 feet.
8. $(0, 0), (-2, 5), (4, 2)$. 16. 5.547, 4.364.
10. $(-1, -1), (5 \mp 2\sqrt{6}, 5 \pm 2\sqrt{6})$. 17. An ellipse.
11. $2x^2 + 2y^2 - 5px = 0$. 18. $16x^2 + 25y^2 = 625$.
12. 4 p.

Page 121

1. (a) $x_1y + y_1x = 2c$. 2. (a) $3x - 2y - 1 = 0, 2x + 3y - 5 = 0$.
- (b) $x_1x + y_1y = r^2$. (b) $3x + 4y = 25, 4x - 3y = 0$.
- (c) $y - 3x_1^2x + 2y_1 = 0$. (c) $2x + y - 7 = 0, x - 2y + 4 = 0$.
- (d) $2y_1y - 3x_1^2x + y_1^2 = 0$.
- (e) $y + y_1 = 2ax_1x + b(x + x_1) + 2c$. 5. $(\pm \frac{a\sqrt{2}}{2}, \pm \frac{a\sqrt{2}}{2})$.
- (f) $ax - 3y_1^2y + 2ax_1 + 3b = 0$. 6. Extremities of the axes.

Pages 127-128

1. $x_1x - 5(x + x_1) + y_1y = 0, x_1y - xy_1 - 5(y - y_1) = 0$.
2. $(5 + \frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}), (5 - \frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}), y = x - 5 \pm 5\sqrt{2}$.
3. (a) $x + y = 2, x - y = 0, \sqrt{2}, \sqrt{2}, 1, 1$.
 (b) $3x + y = 10, x - 3y = 0, \frac{\sqrt{10}}{3}, \sqrt{10}, \frac{1}{3}, 3$.
 (c) $3x - 2y = 5, 2x + 3y = 12, \frac{2\sqrt{13}}{3}, \sqrt{13}, \frac{4}{3}, 3$.
4. (a) $y + 3x = \pm 12$. (b) $2y = 2x + 7$. (c) $y + x = 5 \pm 5\sqrt{2}$.
7. $2y + x \pm 5 = 0, y = 2x \pm 2\sqrt{10}$.

Page 131

1. $2x + 25y = 0$.

2. $x - 2y = 0$.

3. $y = 1, y = 4x + k$.

Pages 133-134

1. $2x + 9y = 9$.

2. $x_1x + y_1y = r^2$.

5. $(6, 1)$.

6. $\left(\frac{c}{a}, -\frac{bp}{a}\right), a \neq 0$.

Pages 143-144. Miscellaneous Exercises

14. $x = a \log \left(\frac{y \pm \sqrt{y^2 - a^2}}{a} \right)$.

15. $y = \frac{e^x \pm e^{-x}}{2}$.

16. $(x^2 + y^2 + c^2)^2 - 4c^2x^2 = a^4$ (x -axis through fixed points with origin midway between them).

17. $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$.

20. $x^2 + y^2 + 20x - 8y = 0$.

19. (a) Ellipse.

21. $py^2 - 2x(x^2 + y^2) = 0$.

(b) Hyperbola.

22. (a) $b^2(x^2 + y^2)^2 = a^2(b^2x^2 + a^2y^2)$.

(c) Oval of Cassini.

(b) $a^2(x^2 + y^2)^2 = b^2(b^2x^2 + a^2y^2)$.

(d) Circle or straight line.

23. $b^2(x^2 + y^2)^2 = a^2(b^2x^2 - a^2y^2)$.

Pages 148-149

15. $x = \frac{2p}{m^2}, y = \frac{2p}{m}$.

16. (a) $x = 3 \cos \pi t, y = 3 \sin \pi t$. (b) $x = -3 \sin \pi t, y = 3 \cos \pi t$.

(c) $x = 3 \cos \pi(t + \frac{1}{4}), y = 3 \sin \pi(t + \frac{1}{4})$. (t is time in seconds.)

17. $x = v_0t, y = -\frac{gt^2}{2}$.

19. 1385.6 feet or 67,896.4 feet.

20. 120 feet per second.

18. 69,282 feet, 10,000 feet.

Page 151

3. $x = a(\theta - \sin \theta), y = a(\cos \theta - 1)$.

4. $x = a(\theta - \pi - \sin \theta), y = -a(1 + \cos \theta)$.

Pages 155-156. Miscellaneous Exercises

1. $2y^2 = x + 1$.

9. $x^2 + y^2 = 2$.

2. $y = 2x^2$.

10. $y^2 = x^2 \tan^2 \sqrt{x^2 + y^2}$.

3. $x^2 + y^2 - 6x + 8y = 0$.

11. $y = x \tan \log_e \sqrt{x^2 + y^2}$.

4. $xy = 2$.

12. $x^2 + 4x^2y^2 - y^2 = 0$.

5. $x^2 - y^2 = 4$.

13. $(x^2 + y^2)^2 - 6(x^2 + y^2) + 8x - 3 = 0$.

6. $y^2 = 4x^2 - 4x^4$.

14. $(x^2 + y^2)^2 - 4ax^2y = 0$.

7. $y^2(1 + x^2) = x^2$.

15. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

8. $2y^2 + 3y = x$.

$$16. 27x^2 = 2(3-y)(3+2y)^2.$$

$$24. x = OA \cos kt + AB \cos 2kt, y = OA \sin kt + AB \sin 2kt.$$

$$28. P: x = 2 \cos kt, y = 2 \sin kt. Q: x = 2(\cos kt + \sqrt{\cos^2 kt + 8}), y = 0.$$

$$29. x = OQ \cos k_1 t + OQ' \cos k_2 t, y = OQ \sin k_1 t + OQ' \sin k_2 t.$$

Pages 159-160

$$18. y = x \tan \frac{a}{\sqrt{x^2 + y^2}}.$$

$$19. (x^2 + y^2)^3 = a^2(x^2 - y^2)^2.$$

$$21. \rho = a\theta.$$

$$20. (x^2 + y^2 + ay)^2 = a^2(x^2 + y^2).$$

Page 161

1. Eight intersections.
2. $(a, 1)$, $(-a, -1)$, and an infinite number of other intersections.
3. $\left(\sqrt{2}, \frac{\pi}{4}\right)$, $\left(\sqrt{2}, \frac{5\pi}{4}\right)$.
4. $(0, 0)$, $(a, 1)$.
7. $(-a, \pi)$ and at the pole.
8. $\left(4, \frac{\pi}{4}\right)$ and at the pole.
9. $\left(a, \frac{\pi}{2}\right)$ and another intersection.
10. $\left(a, \frac{\pi}{3}\right)$, $\left(a, -\frac{\pi}{3}\right)$, and at the pole.
12. $\left(4, \frac{\pi}{2}\right)$, $\left(-\frac{12}{5}, 216^\circ 52'\right)$, and at the pole.
13. $(0, \pi)$, $\left(\frac{a}{2}, \frac{2\pi}{3}\right)$, $\left(\frac{a}{2}, \frac{4\pi}{3}\right)$, and two other intersections.
14. $(0, 0)$, $\left(a, \frac{3\pi}{2}\right)$, and two other intersections.

Pages 164-165. Miscellaneous Exercises

$$9. xy = a(x + y).$$

$$10. y^3 = a^2 x^2.$$

$$13. a\rho = \pm \csc 4\theta.$$

Pages 182-183

1. (a) 6, direction cosines $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$.
(e) 5, direction cosines 0, 1, 0.
2. (a) 7, direction cosines $\frac{6}{7}, \frac{3}{7}, -\frac{2}{7}$.
(f) 10, direction cosines 1, 0, 0.
3. (a) $\gamma = 45^\circ, 135^\circ$.
(b) Impossible.

$$5. 54^\circ 44'.$$

$$6. (a) \frac{1}{3}, \frac{2}{3}, \frac{2}{3}.$$

$$(b) \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0.$$

$$8. \cos^{-1} \frac{4}{5}.$$

$$9. \cos^{-1} \frac{9}{25}.$$

Pages 187-188

2. (a) 2.
(b) $\sqrt{3}$.
6. $3x - 2y = 0$.
7. $5y + 3z = 15$.
8. $2x + z = 13$.
9. (a) $5x - 34y + 18z + 51 = 0$.
(b) $3x + y - 4z = 0$.
11. (a) $\frac{\sqrt{14}}{7}, -\frac{\sqrt{14}}{14}, \frac{3\sqrt{14}}{14}$.
(c) $\frac{4}{5}, -\frac{3}{5}, 0$.
12. $3x + 6y + 2z = 49$.
13. $x - 4y + 3z + 1 = 0$.
14. $x - 3y + 2z = 11$.
17. 60° .
18. $54^\circ 44'$.
19. (a) $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
(b) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.
(c) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$.
20. $\frac{3}{2}$.
22. $(2, -1, 6)$.
24. $(8, 4, 1)$.

Pages 192-193

1. (a) $5x + 4y = 34$,
 $3y + 5z = 8$,
 $3x - 4z = 14$.
2. (a) $(\frac{14}{3}, \frac{8}{3}, 0)$, $(\frac{34}{5}, 0, \frac{8}{5})$,
 $(0, \frac{17}{2}, -\frac{7}{2})$.
3. (a) $\frac{x-6}{8} = \frac{y-2}{4} = \frac{z-4}{1}$.
(b) $\frac{x+1}{6} = \frac{y-3}{-3} = \frac{z+2}{2}$.
(c) $\frac{x-3}{4} = \frac{y-2}{3}, z = 0$.
4. $\frac{x-2}{11} = \frac{y-6}{2} = \frac{z-1}{10}$.
5. $x = y = z$.
8. $\frac{x-5}{2} = \frac{y-4}{3} = \frac{z-8}{6}$.
9. $(-2, 1, 0)$.
10. $\frac{5\sqrt{5}}{2}$.
11. $(3, 5, 3)$.
12. (a) $\frac{x}{6} = \frac{y-\frac{7}{2}}{3} = \frac{z+\frac{7}{3}}{2}$.
13. (a) $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$.
14. $\frac{x+5}{7} = \frac{y-2}{4} = \frac{z+3}{4}$.

Pages 193-194. Miscellaneous Exercises

1. $x + 2z = 9$.
2. $3x + 5y - z + 5 = 0$.
3. $(\frac{5}{3}, \frac{5}{3}, \frac{16}{3})$.
6. $5x + y + z = 7$.
7. $4x - y + 2z = 11$.
9. (a) 4.
10. 5.
11. $2\sqrt{6}$.
12. 30° .
13. $x + 4y + 2z = 1$.
14. $5x - 7y - 4z + 59 = 0$.
15. $\frac{x-6}{2} = \frac{y-4}{2} = \frac{z-1}{1}$,
 $\frac{x-6}{34} = \frac{y-4}{-38} = \frac{z-1}{-1}$.
17. $3x - 5y - 5z + 4 = 0$.
18. $7x + 7z - 4 = 0$.

Page 196

2. $y^2 = 8x - 16$.

Pages 197-198

1. (a) $z^2 + y^2 = 4x$.
 (c) $x^2 + y^2 - (4-z)^2 = 0$.
 (f) $y^2 + z^2 = \sin^2 x$.
4. $\frac{x^2}{64} + \frac{y^2}{64} + \frac{z^2}{289} = 1$.
6. $9x^2 - 16(y-3)^2 + 9z^2 = 0$.
3. $y^2 + z^2 - 2ax + a^2 = 0$.
 7. $(x^2 + y^2 + z^2 - a^2 + b^2)^2 = 4b^2(y^2 + z^2)$.

Page 206

1. $y = \sqrt{3}x$, $4y^2 + 3z^2 = 48$, $4x^2 + z^2 = 16$.
 2. $x^2 = y$, $y + z^2 = 1$, $x^2 + z^2 = 1$.
 3. In xy -plane a circle, in yz -plane an ellipse, in xz -plane a straight line.

Pages 208-209

1. (a) In rectangular coördinates $(2\sqrt{6}, 2\sqrt{6}, 4)$, in cylindrical coördinates $(4\sqrt{3}, \frac{\pi}{4}, 4)$.
 2. (a) In rectangular coördinates $(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 6)$, in spherical coördinates $(\sqrt{61}, \frac{\pi}{6}, \tan^{-1} \frac{5}{6})$.
 3. (a) In cylindrical coördinates $(3\sqrt{5}, \tan^{-1} \frac{1}{2}, 0)$, in spherical coördinates $(3\sqrt{5}, \tan^{-1} \frac{1}{2}, \frac{\pi}{2})$.
 6. (a) $r^2 = 5z$. (b) $z = a \tan \theta$ (cylindrical), $\rho = a \tan \theta \sec \phi$ (spherical).
 7. (a) $(x^2 + y^2 + z^2)(x^2 + y^2) = 6xyz$. (c) $x^2 + y^2 - (z-a)^2 = 0$.

Pages 209-210. Miscellaneous Exercises

2. $(x-3)^2 + (y-4)^2 = 25$.
 3. $9x^2 - 144y^2 - 16z^2 = 0$.
 4. $x^2 + y^2 + z^2 - 14x = 0$.
 5. $5x - y - 4z - 53 = 0$.
6. $(2, 4, 0)$, $(0, 2, 1)$.
 7. $x^2(y^2 + z^2) = a^4$ (about x -axis).
 10. $x^2 + y^2 - (z-2)^2 = 0$.
 11. $16p^2(x-p)^2 + 16p^2z^2 - (4p^2 - y^2)^2 = 0$.

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